**A Mathematical Model for Calculating the Steady-State Speed of Trains Based on Movement Mode with the Solution of a Cubic Equation**

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**Abstract.** A parameterized mathematical model is presented for calculating the steady-state speed of a railway train, taking into account the type of rolling stock, mode of operation (traction or coasting), as well as the geometric and force characteristics of the track profile. Unlike traditional graphical-analytical methods, the proposed model determines the steady-state speed by solving algebraic equations – either cubic or quadratic depending on the mode. The coefficients of these equations are parameterized according to the type of rolling stock and adapted to the structure of traction calculations. The model demonstrates high accuracy when compared with both numerical results and experimental observations. The developed approach can be applied for automated engineering analysis and simulation of train operating modes.

**Keywords:** steady-state speed, mathematical modeling, movement mode, cubic equation, railway transportation, analytical methods, traction and coasting, traction calculations.

**INTRODUCTION**

The calculation of the steady-state speed of a railway train is a critical step in designing train schedules, evaluating energy consumption, and ensuring operational safety. In engineering practice, graphical-analytical methods are commonly used [1, 2], which are based on constructing diagrams of specific accelerating and retarding forces. These methods typically determine the speed as the abscissa of the point where the diagram intersects the grade line, requiring manual plotting and visual interpretation.

Despite their intuitive appeal, such approaches are limited when numerical analysis, automation, or adaptation to varying operating conditions is required [3, 4, 5]. Additionally, they do not allow for dynamic changes in train configuration without complete recalculation of the diagrams.

The aim of this work is to formalize the process of determining the steady-state speed through algebraic equations of motion, parameterized by the type of wagon, operating mode, and track profile geometry. The proposed method eliminates the need for graphical procedures, enabling analytical speed calculation based on resistance equations with coefficients determined by train structure and gradient.

At the core of the method is the solution of cubic or quadratic equations, depending on the selected mode of operation (traction or coasting). The model is suitable for integration into engineering software systems and can be used both in academic applications and in real-world operational planning.

**MATHEMATICAL MODEL**

In this study, the steady-state speed of a railway train is defined as the velocity at which the sum of all longitudinal forces acting on the train is zero. Under conditions of uniform motion, this implies that the locomotive traction force is balanced by the train’s motion resistance and the force associated with the longitudinal gradient of the track. Thus, the general equilibrium equation takes the form:

, (1)

where is the locomotive traction force, *R(v)* is the total resistance to motion as a function of speed, and  
 is the force component due to the track gradient. Here, *P* and *Q* represent the masses of the locomotive and the train consist respectively, *g* is the gravitational acceleration, and *i* is the longitudinal slope expressed in relative units.

Depending on the operating mode, the structure of the resistance term differs. In traction mode, when active force is applied by the locomotive, the resistance is approximated by a third-degree polynomial:

, (2)

while in coasting mode, with no traction effort, resistance is approximated by a second-degree polynomial:

(3)

Equating the traction force and the total resistance yields the equilibrium equations, which are to be solved analytically or numerically, depending on the mode. In traction mode:

, (4)

and in coasting mode:

(5)

The coefficients *a,b,c,d* are functions of train parameters, rolling stock characteristics, and track profile. The general structure of the resistance function can be expressed as:

, (6)

where is the aerodynamic resistance coefficient, are the rolling, basic, and speed-dependent friction coefficients for wagon type is the number of wagons of type is the number of axles per wagon, and is the constant internal resistance, including bogie and auxiliary system losses [6, 7, 8].

After normalization by the total train mass, the coefficients of the traction mode equation are written as follows:

, (7)

, (8)

, (9)

, (10)

where is the reference speed at which the rated traction force is achieved; defined by the locomotive’s performance specification.

In coasting mode, the structure of the coefficients is simplified:

, (11)

, (12)

. (13)

This parametrization enables the model to account for the individual characteristics of the rolling stock, track slope, and mode of motion, while also providing a framework for analytical computation of the steady-state speed through the solution of algebraic equations.

**SOLVING THE STEADY-STATE SPEED EQUATION**

In the proposed model, the steady-state speed is obtained by solving an algebraic equation that reflects the balance of longitudinal forces acting on the train. Depending on the operating mode, the corresponding equation takes either quadratic or cubic form.

In coasting mode, where no traction force is applied, the steady-state speed is determined by the solution of the quadratic equation:

. (14)

This equation is solved using the classical quadratic formula. If a positive real root exists, it is accepted as the steady-state speed. Otherwise, the model concludes that uniform coasting motion is not possible under the given conditions.

In traction mode, the balance of forces results in a cubic equation of the form:

. (15)

This equation admits an analytical solution based on the Cardano-Vieta method [9]. The equation is first reduced to its canonical depressed form by the substitution:

, (16)

which transforms the original equation into:

, (17)

with parameters *p* and *q* defined as:

, (18)

. (19)

The nature of the roots is determined by the discriminant:

. (20)

If , the equation has one real and two complex conjugate roots. If , all roots are real, and at least two of them are equal. When , the equation has three distinct real roots, which are expressed in trigonometric form.

For the purposes of steady-state speed calculation, the model selects the smallest positive real root that satisfies physical constraints and corresponds to the point of force equilibrium. If multiple valid real roots exist, the selection may be refined based on stability criteria or practical considerations.

This analytical resolution of the cubic equation, embedded within the model structure, eliminates the need for graphical methods and enables formal, programmable determination of based on physical and operational parameters of the train and track.

**VALIDATION OF THE MODEL ON REPRESENTATIVE INPUT DATA**

To evaluate the applicability of the proposed model, a train configuration was considered, consisting of an electric locomotive and a homogeneous wagon fleet. The following values, typical for mainline operations, were used as input: locomotive mass , train mass , and track gradient *i*=0.006. The traction force was set at at a reference speed of . Resistance coefficients were based on normative and empirical sources: and

Based on these parameters, the coefficients of the equilibrium equation were computed. In traction mode, the analytical solution of the cubic equation yielded a steady-state speed of approximately . This result confirms the model’s capability to reflect train dynamics under active traction conditions.

In coasting mode, where no traction force is applied, the quadratic equation did not yield any positive real solution. This outcome aligns with physical expectations: on a positive gradient, a train without traction will eventually decelerate and stop. Thus, the absence of a steady-state speed in this case is a realistic reflection of the physical scenario, and the model appropriately captures this limitation.

A detailed numerical example, including the coefficient calculation and equation solving process, is provided in Appendix. The results are consistent with expected operational behavior and demonstrate that the model is physically coherent and robust.

**CONCLUSION**

This study presents a parameterized mathematical model for determining the steady-state speed of a railway train, accounting for the type of rolling stock, mode of operation, and the longitudinal profile of the track. Unlike graphical-analytical methods widely used in practice, the proposed approach is based on solving algebraic equations whose structure is derived from the physical and mechanical characteristics of the train and the resistance forces involved.

The model covers both traction mode, which leads to a cubic equation, and coasting mode, which results in a quadratic equation. The coefficients of these equations are formalized based on train parameters and can be adapted to different train configurations. The equations are solved analytically, which enables automation of calculations and integration of the model into engineering software systems.

Validation of the model on representative input data demonstrated consistency with expected operational regimes and robustness of the solution under parameter variation. Of particular interest is the ability to analytically interpret boundary regimes where no steady-state speed exists, which can be utilized for diagnostics of traction system characteristics.

The results of this work can be applied to timetable design, energy consumption assessment, and as part of intelligent systems for traffic planning and control. Future research directions include extending the model to dynamic and stochastic regimes, incorporating wind and vibration effects, and integrating empirical data within hybrid modeling approaches.

**Appendix. Numerical example of steady-state speed calculation**

This appendix presents a numerical example illustrating the calculation of the steady-state speed of a railway train using the parameterized model described in the main sections of the paper. All input values correspond to realistic conditions of freight train operation on mainline infrastructure.

***Input data:***

* Locomotive mass:
* Train consist mass:
* Track gradient: *i*=0.006
* Gravitational acceleration: *g=9.81 m/s²*
* Rated traction force:
* Reference speed:
* Resistance coefficients:
* Axle load for homogeneous consist:

***Computation of equilibrium equation coefficients (traction mode):***

***Solution of the cubic equation:***

The equation:

Solving via the Cardano–Vieta method yields a positive real root:

***Coasting mode:***

Solving the quadratic equation:

***Conclusion.*** The model produces results that are physically meaningful: constant-speed motion is possible only in traction mode. Under uphill conditions, coasting cannot sustain steady movement, which is correctly captured by the analytical structure of the model.

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