Single Facility Flow Location Modeling to Minimize Total Cost of the Flow

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**Abstract.** The arrangement of facilities within a network is a key factor in optimizing the allocation of resources while reducing operational expenses. We examine a problem involving the selection of an arc where a single facility has to be placed. Our aim is to choose such an arc where total cost will be the least fulfilling demand and supply in a directed graph with cost and capacities. In cases where the flow is not feasible on the specified demand supply requirements, we want to determine the minimum reduction on demand-supply and corresponding minimum cost. We investigate algorithms to solve such problem in polynomial time.

**INTRODUCTION**

In modern society, efficient resource allocation and optimization play vital roles in enhancing productivity and sustainability. The study of network flow with minimum cost emerges as a crucial field in various areas including Applied mathematics, Operations Research and Computer Science. The minimum cost flow problem is the most significant of all network flow problems. Such a problem integrates basic elements of the shortest path problem and the maximum flow problem. Classical minimum cost flow problems are used to determine the most economical method of transporting a specific amount of flow through a provided network without violating the capacity constraints of the utilized arcs.

There are situations in which facilities have to be assigned to the arcs of a network. For example, emergency services such as police stations, fire stations, and health clinics should be placed along the edges of the highway which reduce the capacity of the traffic flow, thereby impacting the associated flow costs. Locating facilities along the arcs of a network presents a significant challenge, incorporating diverse applications in transportation, telecommunications, supply chain management, and urban planning. In this work, we focus on selection of an arc from a given set of arcs of a network to place a facility. When a facility is placed on randomly chosen arc, it may increase the minimum cost. So, we seek such an arc to place a facility so that there is the least increase in minimum cost fulfilling given demand and supply.

**Example 1:** Let's consider a directed graph with given capacities and costs on arcs as shown in Figure 1. Each edge is labeled by two numbers, first represents the capacity of the edge and second denotes the cost per unit flow on the edge. Suppose node 1 is a supply node, providing a quantity of 9 units to the demand node 6. Initially, the value of minimum cost to fulfill this demand is 102 (Figure 2, edge labels represent flow and cost). Now, consider a situation in which a facility of size is to be placed in any one of the arcs in .The minimum cost to satisfy the demand becomes 118 (Figure 3) when the facility is placed on the edge. If the facility is put on the arc (Figure 4), the minimum cost is 116. However, there exists no feasible flow to fulfill demand-supply when the facility is put on the arc. Hence the optimal solution is the selection of arc.

In some situations, placement of a facility on any of the arcs of a set of given arcs to place facilities, the flow is not feasible to fulfill the given demand/supply. In such situations we want to determine the arc to place a facility so that there is minimum reduction in demand/supply and corresponding minimum cost.

In this paper, we develop a network flow model to find the optimal arc, to place a single facility of given size, to minimize the cost of the flow. Designing algorithms to solve such a problem, we analyze the time complexity of the problem. In cases when placement of the facility makes the flow infeasible, we seek an arc to minimize the reduction in demand and supply and then minimizing the cost. To the best of our knowledge, such problems have not been considered in the existing literature.

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| C:\Users\Acer\AppData\Local\Microsoft\Windows\INetCache\Content.Word\fig-1.1.png  **Figure 1:** Network for Example 1 | C:\Users\Acer\AppData\Local\Microsoft\Windows\INetCache\Content.Word\fig-1.2.png**Figure 2**: Minimum cost flow without placing a facility for Example 1 |
| C:\Users\Acer\AppData\Local\Microsoft\Windows\INetCache\Content.Word\fig-1.3.png  **Figure 3:** Min cost flow when the facility is located on the edge (1, 2) for Example 1 | **C:\Users\Acer\AppData\Local\Microsoft\Windows\INetCache\Content.Word\fig-1.4.png**  **Figure 4:** Min cost flow when the facility is located on the edge (2, 5) for Example 1 |

**RELATED WORKS**

Shortest path, maximum flow and minimum cost flow problems are fundamental ideas in the study of network optimization. The shortest path algorithm was first developed by Edsger Dijkstra in 1956. Then Bellman, Floyd-Warshall and many other researchers contributed to new algorithms. Ford and Fulkerson [1] provide the famous Ford-Fulkerson algorithm to find the maximum flow in the network optimization. Edmonds and Karp [2] suggest polynomial time implementation of the labeling algorithm of Ford and Fulkerson. Classical minimum cost problems can be solved by different algorithms such as cycle canceling algorithm, successive shortest path algorithm, primal dual algorithm and many more Ajuja et al [3] These algorithms are most fundamental for the minimum cost flow problem with pseudo polynomial time complexity. Over time, numerous researchers advance these methods to polynomial time algorithms. Goldberg and Tarjan [4] demonstrate cycle-cancellation strategies that result in polynomial-time cycle-canceling algorithms to find minimum cost flow. A strongly polynomial time for the minimum cost flow problem, based on a refinement of the Edmonds-Karp scaling technique is considered as faster algorithm to find the minimum cost [5]. Hu et al [6] present an efficient algorithm for solving minimum cost flow problem with complementary slackness conditions.

Research on the facility location problem has been ongoing research area for several decades. This problem involves identifying the most efficient locations for facilities such as warehouses, distribution centers, and service centers to optimize both flow and cost, taking into account factors such as transportation costs, demand and supply patterns, and network topology. In the 20th century, a classical location theory establishing the foundation for subsequent research in facility location, with the p-center and p-median problems serving as fundamental principle are developed [7]. A flow location problem is a type of network flow problem that helps in deciding where to place given facility. Daskin [8] presents a comprehensive treatize of the literature on facility location problems. Kube and Lim [9] introduce location for alternative fuel station using the flow-refueling location model and dispersion of candidate sites on arcs. Hamacher et al [10] introduce the facility location problem which aims to identify arcs where maximum flow reduction is minimized. Nath and Dhamala [11] develop a network flow approach for locating optimal sink in evacuation planning. Dhungana and Dhamala [12] combine network facility placement and contraflow technique to reduce flow loss in evacuation networks. Scheiper and Schiffer [13] introduce an approach that merges a model for determining charging station locations with a power flow model incorporating integrated energy storage. Nath et al [14] present a problem of optimal facility location in such a way that increase in the quickest time of contraflow is minimized. FlowLoc problems with maximum excess flow to minimize the decrease in the maximum flow within an evacuation network, where predefined facilities are situated along the edges, and any surplus flow is stored at nodes are presented in [15].

**PRELEMIAARIES**

To keep the paper self-contained, we present the fundamental concepts of network optimization. For comprehensive treatise of these procedures, we refer to [3]. A network is a directed graph G with a set of nodes V and set of arcs A, such that and. Associated with each node is a real number such that. A node is called a supply node, a demand node or a transshipment node according as or .

Associated with each arc is a non-negative capacity and the cost. We denote the vector by and by known as supply and cost respectively. Let , representing a network with vertex set , arcs set , supply , cost and capacity . A vector is called flow on the network if it satisfies

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|  | (2) |

The quantity represents something that flows from node to node. A feasible flow fulfills the supply/demand requirements at each node without exceeding the arc capacities

**Residual Network**

In solving various network flow optimization problems, a residual network has to be constructed. Given a flow in , the residual network contains same set of nodes as those in . For each , contains the arc with residual capacity with cost and an arc with residual capacity and cost. Moreover, contains only arcs with positive residual capacity.

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| C:\Users\Acer\AppData\Local\Microsoft\Windows\INetCache\Content.Word\fig-1.5.png  **Figure 5:** Network | C:\Users\Acer\AppData\Local\Microsoft\Windows\INetCache\Content.Word\fig-1.6.png  **Figure 6:** Residual network of |

Flow augmentation along the arc in by units reduces by units and increases by the same.

**Maximum Flow Model**

Given two special nodes and, called the source node and the sink node in the network , with , the maximum - flow problem is set to maximize . Hence, the problem can be modeled as:

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The quantity is called the value of the flow.

**Minimum Cost Flow Model**

The minimum cost flow model is a classic optimization problem in the field of network flow. It is used to find the most economical way to send materials or goods through a network, such as communication networks, supply chain networks or transportation networks. The problem can be stated as follows:

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where .

The minimum cost flow problem relies on the assumption that it has a feasible flow that satisfies the demand/supply requirements capacity constraints. Moreover, most of the algorithms to solve the minimum cost flow problem start with the feasible flow. The procedure to check whether there exists a feasible flow as outlined in [3] is presented in the Algorithm 1.

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| **Algorithm 1** Feasibility of Flow for Minimum Cost |
| Input: directed network  1:  2.  3. s.t. s.t. bi < 0  4: find the maximum flow in the network.  5: if s.t. and s.t. then  6: , restricted to , is a feasible flow in  7: else  8: ßthere does not exist a feasible flow in  9: end if |

To solve minimum cost flow problem, various algorithms and approaches have been developed. Some of the key algorithms are: cycle canceling algorithm [16], successive shortest path algorithm [17], primal-dual algorithm [18], out of kilter algorithm [19], relaxation algorithm [20], cost scaling algorithm [21] capacity scaling [22] and minimum mean cycle canceling algorithm [4]. Among the algorithms to find the minimum cost, a simple-to-describe strongly polynomial running time algorithm is minimum mean cycle-cancelling algorithm idea of which is used to develop Algorithm 5 and Algorithm 6 in our work. The mean cost of a directed cycle in W in G is . The algorithm adapted from [3] is presented in Algorithm 2. It is assumed that there exists a feasible flow to meet the supply/demand requirements.

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| **Algorithm 2** Minimum Mean Cycle Cancelling Algorithm |
| **Input:** directed network graph  **Output**: minimum cost flow  1: establish a feasible flow x in  2: construct residual network  3: find a negative cycle W in G(x) with minimum.  4: the minimum residual capacity along  5: update the flow x by augmenting amount of flow along  6: repeat steps until no negative cycles are found  7: return |

Starting with a feasible flow x, the algorithm iteratively augments the maximum possible flow along the minimum-cost mean cycle with negative cost in the residual network G(x). When no negative cycle exists in the G(x), the algorithm terminates with the minimum cost flow x. The running time of the algorithm is [3].

**SINGLE FACILITY FLOWLOC TO MINIMIZE THE TOTAL COST**

In this section, we consider a problem of placing a facility on an arc which reduces the capacity of the arc resulting in the increase minimum cost of the flow. From a set of given arcs, we choose such an arc so that the increase in the minimum cost is the minimum because of the placement of the facility. Let r be the size of the facility and be the set of arcs available for placing the facility. The problem can be formulated as:

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where,

Constraint (10) represents the mass-balance condition. Constraint (11) relates with capacity reduction of the arc on locating the facility. Constraint (12) is capacity constraint on . Constriant (13) represents that a facility must be placed on only one edge. Constraint (14) represents that: if facility is located on the arc then otherwise 0. A simple iterative procedure to solve the problem is given in Algorithm 3.

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| **Algorithm 3** Single Facility FlowLoc to minimize the Total Cost (I) |
| Input: directed network G = (V, A, b, c, u), set of locations L ⊆ A, facility size r  Output: location to place the facility, minimum cost, minimum cost flow  1: loc none, mincost , none  2: **for** **do**  3:  4: test whether a feasible flow exists in using Algorithm 1  5:  **if** a feasible flow exists **then**  6: minimum cost flow in  7: mincost temp total cost of x  8:  9: **if** mincost mincost temp then  10: mincost mincost temp  11: loc  12: **end if**  13: **end if**  14: **end for**  15: **return** loc, mincost, |

Algorithm 3 iterates through all possible locations and find the minimum cost. If location hosts the facility and its minimum cost is the lowest compared to all other minimum cost values while locating the facility at other locations, then l is the optimal location.

**Theorem 1.** *The time complexity of the Algorithm 3 is where and are the time complexities of the maximum flow and minimum cost flow computations.*

*Proof.* The algorithm 3 (line 4) first checks feasibility of the flow on placing the facility by finding maximum flow whose complexity is MF, then it finds minimum cost flow, the complexity of which is MC, if there is a feasible flow. Since it iterates over all , the time complexity of the algorithm in worst case is □

**Theorem 2***. The single facility FlowLoc problem to minimize the total cost can be solved in strongly polynomial time in m, n.*

Proof. The maximum flow problem can be solved in strongly polynomial time. For example, using pre-flow push algorithm [3], the problem can be solved in time. The time complexity solving minimum cost flow is also strongly polynomial. For example, the minimum mean cost cycle cancelling algorithm to find the minimum cost flow [22] runs in . Hence the time complexity of the single facility FlowLoc to minimize the total cost of the flow is , which is strongly polynomial in n and m, as. □

If we compute the minimum cost flow x without reducing the capacity of any of the arcs in L and find that for some , then placement of the facility in will not reduce the total cost of the flow and such will be the optimal location. If the residual capacity is not large enough to host the facility then algorithm 3 is used. In such a case, the running time of the algorithm 3 can be significantly improved. Algorithm 4 addresses such a case.

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| **Algorithm 4** Single Facility FlowLoc to Minimize the Total Cost (II) |
| **Input:** directed network, set of possible locations , facility size r  **Output:** location of the facility, minimum cost, minimum cost flow  1: minimum cost flow in G  2: **for** **do**  3: **if** then  4: mincost total cost of x  5: loc  6: **return** loc, mincost,  7: **stop**  8: **end if**  9: **end for**  10: Algorithm 3 |

**SINGLE FACILITY FLOWLOC TO MINIMIZE THE TOTAL COST WITH MINIMUM REDUCTION IN DEMAND/SUPPLY**

In some situations, there may not exist a feasible flow after placing a facility in any of the arcs in L. However, if demand/supply requirements are reduced, we may find the feasible flow for reduced demand/supply. In such cases, it is justifiable to place the facility on such an arc that minimizes the reduction in demand/supply

and also minimizes the total cost of the flow. When there is no feasible flow, we will have for some such that in Line of Algorithm 1. This hints the reduction of the supply at by . However, such a flow x may not yield the minimum cost and

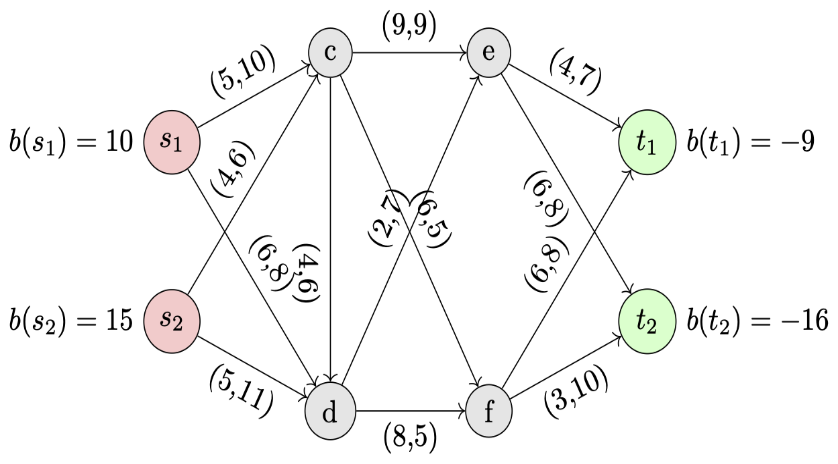
the value of supply reduction at i may be different. Applying the procedure of minimum mean cycle cancelling, we can transform x to have minimum cost with the minimum reduction in supply /demand and identify the reduction in supply at various nodes. We construct Algorithm 5 for the procedure. It is assumed, as before, that.

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| **Algorithm 5** Minimum Cost Flow With Minimum Reduction in Supply/Demand |
| **Input:** directed network  **Output:** minimum cost flow with minimum reduction in  revised b  1:  2:  3: s.t. , and s.t.  4: find the maximum flow in the network.  5: value of  6:  7: **while** there is a negative cycle in the residual network (x) **do**  8: find a negative cycle with minimum mean cost  9: the minimum residual capacity of  10: update by augmenting amount of flow along  11: **end while**  12: **for** do  13: if , then  14: , then  15: **end for**  16: **return** |

**Theorem 3**. *The time complexity of Algorithm 5 is .*

. After adding two extra nodes , and arcs, the algorithm performs the procedure of minimum mean cycle cancelling algorithm (Algorithm 2) running time of which is So the complexity of the Algorithm 5 is which is. □

**Example 2**. Let us consider a network as shown in Figure 7. Nodes and supply and units of goods respectively through the network to the demand nodes and . Node is seeking units where as node requires units of goods. In this network, before placing a facility on any one of the arc minimum cost to distribute the goods from supply nodes to demand nodes is . Now a facility of size has to be located on any one of the arcs set . When the facility is placed along any of the arcs of set L, a feasible flow cannot be achieved.



**Figure 7:** Network for the minimum cost flow

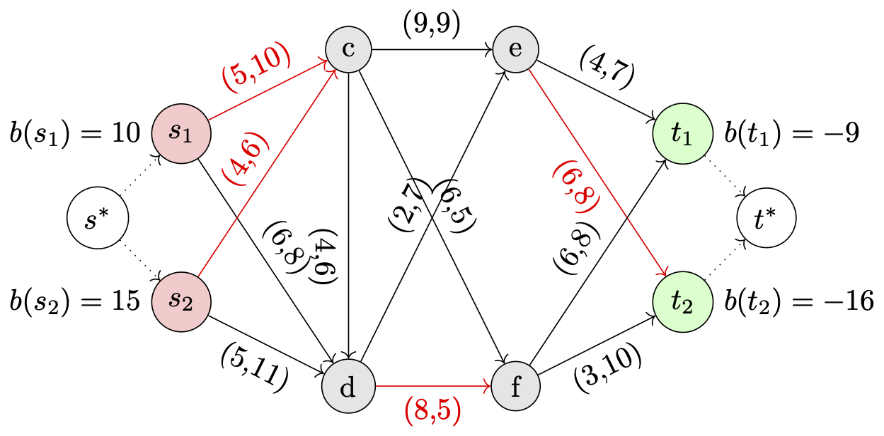
If the facility placement of the facility in an arc results in reduction in (positive) supply, we choose such an arc with minimum reduction in supply and then which minimizes the total cost in lexicographical order. We denote such an order relation by defined as

We propose Algorithm 6 to identify the flow location to minimize the total cost with minimum reduction in demand and supply.

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| **Algorithm** 6 Single Facility FlowLoc to Minimize the Total Cost with Minimum reduction in Supply/Demand |
| **Input:** directed network , set of locations , facility size r  **Output:** location of the facility, minimum cost flow, minimum cost, reduction in  supply/demand , revised  1: loc none, none  2: loc none  3: **for** in L **do**  4:  5: temp (reduction in supply/demand, minimum cost), minimum  cost flow, revised using Algorithm 5  6: , retain  7: **if** temp **then**  8: temp  9: loc  10: **end if**  11: **end for** |

Algorithm 6 iterates through all possible arc by putting the facility. It finds the reduction in demand/supply and corresponding minimum cost. Reduction in demand/supply and corresponding minimum cost are put in order pair. It compares all order pairs found during the iterations by lexicographic way and returns the location for the facility, reduction on demand/supply, minimum cost and minimum cost flow.

**Example 3.** The solution of the problem stated in Example 2 is illustrated in this example.



**Figure 8:** Transformed network related to algorithm 6

It is observed that when the facility is located (Figure 8) on the arc the demand-supply decreases by only one unit, which is least degradation among all. Though the minimum cost value on placing a facility on others arcs is lower, demand /supply is least degraded while placing facility on the arc . So optimal location is and corresponding minimum cost is . The Table 1 given below shows the analysis of reduction on demand/supply and corresponding minimum cost while placing a facility on the given set of arcs.

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| **Facility Placed on arc** | **Reduction on demand/supply** | **Minimum Cost** |  |
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**Table 1:** Analysis of reduction on demand/supply and minimum cost

**Theorem 4**. *If all input data are integers, then the minimum cost flow with facility* *location problem always yields an integer solution.*

Proof. The algorithm initially establishes a feasible flow in the network by solving a maximum flow problem. If arc capacities are integers, the maximum flow problem gives an integer solution [3, p. 186]. Given that costs and the facility size are integers, the reduced capacity is also an integer. Therefore the minimum cost flow problem also has an integer solution [3, p. 318]. The total cost is calculated as the product of flow and cost. Since the product of two integers is also an integer, hence the minimum cost obtained is indeed an integer. □

**CONCLUSION**

In this paper, we studied facility location, maximum flow, shortest path, and minimum cost flow problem. Additionally, we studied several algorithms along with their computational complexities. Our main focus was on introducing a new approach to a single facility location concerned with minimum cost flow. We developed a model for the minimum cost flow by placing a facility along the arcs. We addressed the problem in two situations: first, assessing the feasibility of flow given a demand/supply configuration, and second, determining the least degradation in demand-supply and subsequently finding the corresponding minimum cost flow if the flow is not feasible initially. For both scenarios, we developed new algorithms focused on minimizing the total increase in minimum cost. Furthermore, we analyzed the time complexities of these newly developed algorithms.

This result has theoretical and practical importance for addressing the problem related to minimizing transportation cost and proper facility utilization in developing countries including Nepal. Such kinds of issues often arise regarding the optimal placement of essential facilities such as emergency security, fire stations, and health clinics, focusing to minimize transportation costs. Our solution may help to take right decision for local government or concern departments. We proposed new model and algorithms for a single facility location. Further, we are interested in extending the results to multiple facilities cases as well as in dynamic network topology.

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