Maximum Multi-Commodity FlowLoc for Single-/Multi- Facility

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**Abstract.** An optimal solution to the maximum multi-commodity flow problem in a capacitated network is to send as much feasible flow of more than one commodity from their respective sources to corresponding sinks without violating the capacity constraints. The flow location (FlowLoc) problem concerns with the minimization of reduction of the maximum flow value by fixing the given facility on an appropriate prescribed location. In this paper, we integrate the concepts of FlowLoc and multi-commodity flow problems, introduce the maximum static multi-commodity FlowLoc problems for single and multiple facilities and propose their flow models by taking the prioritized multi-commodities. We solve the single facility problem polynomially. In the case of multi-facility, the polynomial time heuristic is presented to obtain the near optimal solution.

***Keywords****: Multi-commodity flow; FlowLoc; -complete; heuristic*

# Introduction

**Motivation.** The maximum static multi-commodity flow (MSMCF) problem in a transportation network topology involves maximizing the simultaneous movement of various types of commodities, such as people, goods, etc., from their respective origins (sources) to their destinations (sinks) without exceeding capacity constraints. Also, the flow location (FlowLoc) problem focuses on placing a designated facility at the optimal location to minimize the impact on the network's maximum flow value. The mathematical model that combines these problems is very beneficial in the transportation network, to optimize the movement of commodities from their respective sources to their corresponding sinks by optimally positioning the given facility.

**Literature review.** The concept for the MSMCF problem was introduced in [9]. Tomlin [18] formulated the minimum cost multi-commodity network flow problem. The linear and non-linear problems were discussed in the survey paper [3]. The computational comparisons on the multi-commodity network flow algorithms: price directive decomposition, resource directive decomposition and primal partitioning are found in [2]. The multi-commodity flow problem and its flow model are discussed in [1]. Hall et al. [11] solved the MSMCF problem polynomially using ellipsoid or interior point method. The multi-commodity flow formulations of the capacitated vehicle routing problem are found in [16].The survey papers [4,6] provides the different network flow models and their applications. The quickest multi-commodity flow problem with partial lane reversals was solved in [5]. Using proportionality and flow dependent capacity shearing techniques, the MSMCF problem was solved in [14]. The MSMCF problem with intermediate storage was solved polynomially in [13]. The generalized maximum static, maximum dynamic and earliest arrival multi-commodity contraflow problems were introduced and solved by Gupta et al. [10].

The maximum FlowLoc (MFL) problem maximizes the flow value in a network by fixing the prescribed facility in an optimal location. Integrating the concepts of the network flows and location analysis, Hamacher et al. [12] introduced the MFL problems. They solved 1- FlowLoc problem by polynomial time algorithm. In the case of multiple flow location (q-FlowLoc), they presented polynomial time heuristics to obtain the near optimal solution. The single facility ContraFlowLoc problem was introduced and solved polynomially by Dhungana and Dhamala [8]. Nath et al. [17] introduced the quickest FlowLoc (QFL) problem and solved by presenting the polynomial time algorithm. The maximum static and dynamic FlowLoc problems with intermediate storage were introduced by Dhamala et al. [7]. They presented polynomial time algorithms and heuristics in the case of single and multiple facilities, respectively. Wagle et al. [19] introduced the lexicographic maximum static and dynamic FlowLoc problems. They presented the polynomial time algorithms to solve the single facility problems. To obtain the near optimal solution for the case multiple facilities, the polynomial time heuristics were presented. Recently, the maximum dynamic FlowLoc, earliest arrival FlowLoc and ContraFlowLoc problems were introduced by Wagle et al. [20]. Using the temporally repeated approach. They solved the problems polynomially.

**Research Gap.** The MFL and MSMCF problems and their mathematical models with solutions can be found in literature. But, the combination of these problems into a unified approach is a major research gap. To fill this gap, the maximum static multi-commodity 1-FlowLoc (MSMC1FL) and the maximum static multi-commodity q-FlowLoc (MSMCqFL) problems are proposed and solved.

**Our Contribution.** The MSMC1FL and MSMCqFL problems are introduced for the first time. We solve the MSMC1FL and MSMCqFL problems by the polynomial time algorithm and the polynomial time heuristic, respectively.

**Organization of the Paper.** In Section 2, we provide the key terms used throughout the paper. We introduce the MSMC1FL and MSMCqFL problems, provide their mathematical models and solve in Section 3. Section 4 concludes the paper.

## PRILIMINARIES

Let be a dynamic network, where and are the set of nodes and arcs with and , respectively. Suppose that {commodity-1, commodity-2, …, commodity-j} be the set of the given commodities with priority order commodity-1 commodity-2 … commodity-j (i.e. the commodity 1 has the first priority and so on) in . For each , the nodes  and are the source and sink, respectively. For each arc is the capacity function that bounds the flow of commodities. Let and be the sets of incoming arcs to and outgoing arcs from the node , respectively. Suppose that be the set of locations for a given set of facilities **P** of size and then the network is denoted by .

If be the function that defines the maximum number of facilities which can be fixed on every , then the FlowLoc problem is to deal the allocation for all on so that the flow is maximized in the network (i.e. reduced network), where max . If we place more than one facility on an , then is decreased by the size of the largest facility.

**Prioritized approach.** Let commodities are given with the priority order as commodity-1 commodity-2 commodity-j and be the static flow of the commodity on the bundle arc . For the commodity , let be the residual capacity of . Due to the prioritization of the commodities, the flow of commodity on be, min, where be the flow of commodity at . This approach is also discussed in [11]. The commodity prioritized MSMCF problem is introduced and solved in [15]. Note that, if two commodities have the same priority, their priority order is assigned arbitrarily.

In this paper, we maximize the flow by using prioritized approach for the given commodities at the bundle arcs.

**THE MULTI-COMMODITY FLOWLOC**

In this section, MSMC1FL and MSMCqFL problems with mathematical models are introduced. The MSMC1FL problem is solved polynomially and for MSMCqFL problem, the polynomial time heuristic is presented.

**The 1-FlowLoc Problem**

**Problem 1.** *Let be a given network with the set of the prioritized commodities . The MSMC1FL problem maximizes the flow of the commodities from their respective sources to the corresponding sinks by fixing the given facility on an optimal location.*

**Mathematical model.** Let be a given network with static flow function for the prioritized commodity and

The MSMC1FL problem maximizes Objective (1) with respect to Constraints (2) – (3).

Subject to,

Where, Constraints (2) and (3) are the flow of conservations and the capacity constraints, respectively.

To solve the problem, first of all, we fix the given facility on a location and obtain the reduced network . Now we decompose the multi-commodity flow problem to the independent single commodity sub-problems (ISCSP). We solve the maximum static flow (MSF) problem for each prioritized commodity in After this we remove from (so that receive its original capacity), fix on another location , and repeat this process for all remaining location. From this, a sequence of the maximum flow values is obtained. Finally, we take the network with the maximum flow value among these solutions. Here, we present Algorithm 1 to solve the problem.

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| **Algorithm 1:** MSMC1FL algorithm |
| **Input :** Givennetwork location size of the facility is  **Output:** MSMC1FL solution and the optimal location   1. Neglect if 2. Fix on the reduced network is . 3. Reduce the multicommodity flow problem into ISCSP. 4. Compute the prioritized MSMCF in . 5. Remove from fix on , repeat the Steps (1)- (3) and continue to each remaining . 6. Pick up the solution with overall maximum flow value. |

Here, using Algorithm 1 to get the optimal flow value, we need to iterate each . To improve this running time, we present Algorithm 2. Using Algorithm 2, first of all, the problem is decomposed to ISCSP and computed MSMCF in The residual capacity of each is calculated and checked whether there is any whose residual capacity is at least . If such a location is found, is fixed there and calculated the prioritized MSMCF. Otherwise, we apply Algorithm 1 to get the optimal flow. Although the worst-case complexities of both algorithms are same, in practical scenarios, if a location with sufficient residual capacity is available to host the facility, placing the facility there the optimal reduced network is easily obtained. And hence, by maximizing the flow in the reduced network, we can get the optimal solution. From this, the number of max flow calculation is reduced.

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| **Algorithm 2:** Improved MSMC1FL algorithm |
| **Input:**  Network location size of the facility is  **Output:** MSMC1FL solution and the optimal location   1. Reduce the multicommodity flow problem into ISCSP. 2. Compute the prioritized MSMCF in 3. Obtain the residual capacity 4. If with then fix on . The reduced network be 5. Compute the prioritized MSMCF in 6. If apply Algorithm 1. |

Theorem 1. *The optimal solution to MSMC1FL problem can be obtained in polynomial time*.

*Proof.* First of all, the feasibility of Algorithms 1 and 2 is proved. The allocation of the facility, obtaining the reduced network, decomposition of the problem into ISCSP and computation of prioritized MSMCF in the reduced network are feasible*.*

Using Algorithm 2, after decomposition to ISCSP, we compute the prioritized MSMCF in and calculate the capacity for every . If an with is found, then we fix there and compute the prioritized MSMCF in the network .

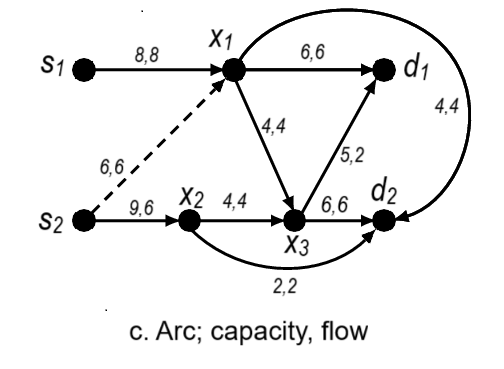
If such a location does not exist, then we place on a location and compute the prioritized MSMCFF by decomposing the problem into ISCSP. This process is repeated for every remaining . Each time, we get the maximum solution. Finally, the maximum flow value overall these solutions is picked up which is optimal.

Here the time complexity to obtain the MSMCF in is , where be the total number of commodities. Since, we can construct the reduced network in linear time, so the optimal solution to the MSMC1FL problem can be obtained in .

**Example 1.** *Consider be a given network with two commodities with priority order commodity-1 commodity-2 having corresponding sources and , and sinks and , respectively, as shown in Figure 1(a). Each arc has a given capacity, for example: has capacity 8. Suppose that a given facility with size should be fixed on one of the locations so that the maximum flow of the commodities can be computed towards their corresponding sinks. For this, at first, the prioritized MSMCF is computed in . Using the paths: and , we can push 8 and 12 units of flow of the commodities 1 and 2, respectively, towards and . The residual capacity of each*

*Since, , so we place on and obtain the optimal reduced network (c.f. Figure 1(b)). Using the paths and , we can send 8 and 12 units of flow of the commodities 1 and 2, respectively, towards and (c.f. Figure 1(c)). Hence, there is no reduction of the maximum flow*.

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**Figure 1**: (a) Given network, (b) Reduced network with optimal location, (c) MSMCFL solution

**Example 2.** *Let in Example 1, then . Also is greater than and , so we neglect and from . By fixing the facility on and the maximum flow of the commodities 1 and 2 towards and , respectively, are 1 and 12, 8 and 9, and 8 each. Thus, the optimal location is and the MSMC1FL solution is the flow of 8 and 9 units of the commodities 1 and 2, respectively.*

**The q-FlowLoc problem**

**Problem 2.** *Let be a given network with the set of the prioritized commodities . The MSMCqFL problem aims to maximize the flow of the commodities from their respective sources to the corresponding sinks by optimally placing the given facilities on the given location(s).*

**Mathematical model.** Suppose that be the given network with the given set of facilities **P**, of size and the decision variable

Then the mathematical model of Problem 2 is the maximization of Objective (1) with respect to Constraints (2) and (4) – (8).

Constraints (4) bound the number of facilities on any . By Constraints (5), each can be fixed on exactly one . Constraints (6) and (7) are the capacity constraints.

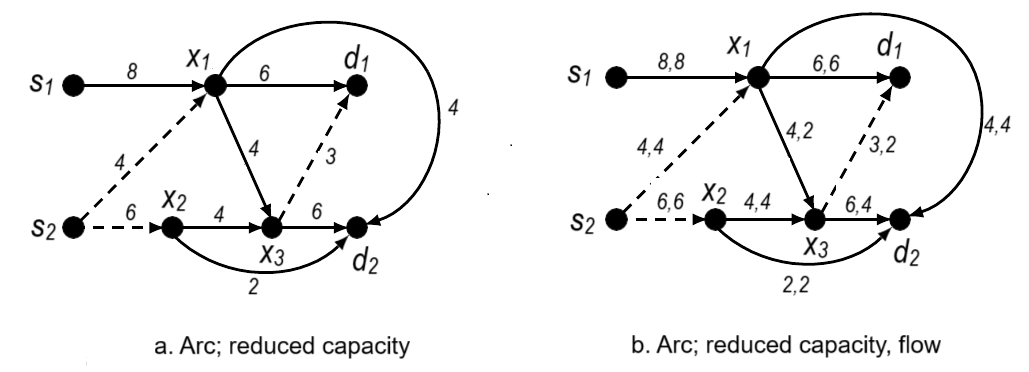
**Theorem 2.** *The MSMCqFL problem is complete.*

*Proof.* The MSMCqFL problem focuses on maximizing the flow of *q* commodities from their respective sources (*sᵢ*) to their destinations (*dᵢ*) by placing *q* facilities. We can reduce this problem by the maximization of the j independent single commodity flow problem by fixing q facilities (i.e. j times single commodity qFlowLoc sub problems). As in Hamacher et al. [12] the single commodity qFlowLoc problem can be reduced to 3-SAT problem. Thus, the problem is Complete. □

To obtain the near optimal solution, a polynomial time heuristic is presented (c. f. Algorithm 3). At first, in the original network , the prioritized MSMCF is computed. The residual capacity of each is calculated and sorted according to the capacity. The facilities are sorted with their size. Now, with the largest residual capacity is selected, placed facilities there and to be continued until all are placed on . During placing facilities on the prioritized , if we find , then we place in and so on. Also, we neglect the facility if its size is greater than of each remaining Finally, the prioritized MSMCF is computed in

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| **Algorithm 3:** MSMCqFL algorithm |
| **Input:** Givennetwork location set of the facilities **P**  **Output:** MSMCqFL solution   1. Compute (1) - (3) of Algorithm 2 and obtain the residual capacity of each . 2. Sort the locations with their residual capacities: . 3. Sort the facilities according to their size: . 4. Place on and so on. 5. Let the prioritized should be fixed on and then fix in . 6. Neglect if for each remaining . 7. Obtain the reduced network . 8. Compute the prioritized MSMCF in . |

**Example 3.** *Suppose that we want to fix the facilities of size and on the location with in the network (c.f. Figure 1(a)) and maximize the static multi-commodity flow value. As in Example 1, at first, the prioritized MSMCF is computed in the original network and calculate the residual capacity of each . The location are sorted according to their . Also, the facilities are sorted with their size: . Using Algorithm 3, we fix , and on and , respectively, and obtain the network (c.f. Figure 2(a)). Finally, in, the prioritized MSMCF is computed (c.f. Figure 2(b)).*



**Figure 2**: a. Reduced network, b. MSMCqFL solution

**CONCLUSIONS**

The MMCF problem is to maximize the flow of the given commodities from their respective sources to the corresponding sinks. On the other hand, the MFL problem concerns with the maximization of the source-sink flow value by placing the prescribed facilities on appropriate places. However, by fixing the facility on an arc, the arc capacity is reduced that may affect the maximum flow value. However, it is necessary to provide the facility to the commodity.

This paper introduces flow models of the MSMC1FL and MSMCqFL problems by incorporating these concepts. Polynomial-time algorithms and heuristic have presented in the single- and multi- facility cases, respectively. These solutions can be useful in transportation networks, especially supplying goods like oil, water, etc. The study's limitation is that the flow should be smooth and prioritized. Also, the cost of placing the facility and any cost of flow are not considered. Our next direction is to solve the problem by considering the flow cost on any arc and the additional cost of establishing the facility at the location.

**Conflict of interest.** The authors confirm there are no conflicts of interest related to the publication of this paper.

**Data availability.** No additional data are utilized by the authors in this article.

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