**Comparative Analysis on the Fractional-Order Epidemic Model of Measles by Various Semi-Analytical Methods**

Sonali Mallik,1, a) Sunita Chand2, b)

***1,2****Department of Mathematics, Siksha ‘O’ Anusandhan Deemed to be University, Bhubaneswar, Odisha 751030, India*

*1Email:*[*malliksonali2003@gmail.com*](mailto:malliksonali2003@gmail.com)

*2Email: sunitachand@soa.ac.in*

**Abstract.** In this paper, the Caputo fractional-order epidemic model for Measles infection has been extensively analyzed and studied, which incorporates memory effects. Several semi-analytical methods such as the Adomian Decomposition Method (ADM), the Differential Transform Method (DTM) and the Laplace Adomian Decomposition Method (LADM) have been applied to solve the model and the results obtained using MATHEMATICA software. Further, the results have been compared numerically and graphically for different parameters of the disease model. Furthermore, the impact of different fractional orders has been explored and compared graphically in view of the disease progression and recovery.

**Keywords:** Caputo fractional derivative, ADM, DTM, LADM, Measles infection.

**INTRODUCTION**

Fractional calculus is the application of mathematical analysis, and it provides several applications in different areas of physics, mathematics, and engineering. Fractional Differential Equations (FDEs) are more versatile and powerful than Ordinary Differential Equations (ODEs) for modeling various real-life applications due to their hereditary and memory-related effects. Podlubny [1] has provided some applications and solutions for FDEs.

A semi-analytical method was introduced by George Adomian between the 1970s and 1990s, which was named after him and is known as the Adomian Decomposition Method (ADM). Agom and Badmus [2] have discussed the ADM to solve the Nonlinear Ordinary Differential Equations (NDEs), which has the same compatibility with the exact solutions. Shawagfeh [3] discussed that several equations of FDEs are solved by ADM. Momani and Shawagfeh [4] have presented that ADM provides more accurate solutions in real-life applications and also uses the Pade approximant to get a successful series solution. Khaled and Momani [5] discussed that ADM is a more efficient method to find the exact solutions for diffusion and wave solutions. Jafari and Gejji [6] presented that the system of FDEs is solved by ADM, and it is free from rounding off errors.

Hussein [7] has studied the LADM with different FDEs. Ongun [8] has studied the LADM to solve the HIV infection model to show its suitability for solving Nonlinear systems of equations. Farman et. al [9] have discussed both the LADM and DTM to get the approximate solution of the disease model and also compared the model for different values of the fractional order. Budiman and Salim [10] have discussed whether the LADM gives the same results as the exact solution or not. Al Sulami et. al [11] have studied the -LADM for the class of  Caputo differential equations with various initial and boundary conditions.

The DTM is a semi-analytical method that can obtain the numerical and analytical solution of integral differential equations (IDEs), differential algebraic equations, and neural delay differential equations. Munganga et. al [12] have studied the DTM and multi-step differential transform method (MSDTM) to solve the initial value differential equations. Mirzaee [13] has discussed that DTM is a semi-analytical method used to solve the linear and nonlinear systems of ODEs to get more converging series solutions. Patil and Khambayat [14] show that DTM is a very reliable application to find the approximate solution, and it is also compared with the Laplace transform method. Hassan [15] presents that DTM results show better agreement with the exact solution and the Runge Kutta method. Saeed and Rahman [16] have discussed that DTM is not only used for linear or nonlinear ODEs but also used to solve systems of delay differential equations (SDDEs). Delay Differential Equations (DDEs) are applied in various fields such as biological population management, scientific fields, neural networks, and so on. Liu et. al [17] have discussed DTM to solve DDEs and also check the feasibility and efficiency of DTM. Bayram et. al [18] have executed the Fractional Differential Transform Method (FDTM) to fractional differential- algebraic equations (FDAEs), also compared FDTM and the Homotopy Analysis Method (HAM) with the exact solution.

Measles is an infectious disease caused by Paramyxovirus. It is also known as Rubella or morbilli. The incubation period is 14 days. The symptoms are fever, rash, cough, and coryza. The nations mostly affected by measles are Pakistan in Asia and Nigeria. It occurs in all the seasons. In Nigeria, every year over 3 million people are saved from this disease due to vaccination. The MMR vaccination can protect against measles. Almeida et. al [19] presented some real-life applications which is solved by FDEs. Nisar et. al [20] have considered the fractional epidemic models of various diseases and checked the equilibrium point and stability of some models. Seidu [21] has discussed the stability, cost-effectiveness and mathematical analysis of malaria disease. Guo et. al [22] have discussed the stochastic model of the measles virus, checked the uniqueness, existence and compared quantitatively the analytical and graphical results. Belay and Alemeh [23] have analyzed and formulated the transmission dynamics of measles and also checked the stability, sensitivity conditions. Akuka et. al [24] have studied mathematical analysis and modeling of the measles model and also used qualitative techniques to establish the existence and uniqueness of the model.

The objective of our work is to perform a comparative analysis of the measles model of fractional order by using ADM, LADM, and DTM, as well as compare the behaviour of the solutions by changing the fractional orders.

Our work is motivated by Agom and Badmus [2], Jafari and Gejji [6], Hussein [7], Budiman and Salim [10], Mirzaee [13], Saeed and Rahman [16], Bayram et. al [18], Seidu [21], Akuka et. al [24].

This paper has been presented as follows. Section 1 consists of the introduction part, and section 2 contains the preliminaries and basic definitions relevant to our work. Section 3 consists of the model formulation, and Section 4 gives the methodology used, which involves solving the model with the help of ADM, LADM, and DTM. Section 5 gives the numerical results and section 6 contains the results and discussion. Section 7 contains the conclusion of the work.

**PRELIMINARIES**

**Riemann-Liouville Fractional Derivative [1]**

The Riemann-Liouville Fractional Derivative of order  of a function is defined by  (1)

**Caputo Fractional Derivative [1]**

The Caputo Fractional Derivative for  of a function is defined by

 (2)

**Fractional Integral [1]**

The fractional integral is defined as

 (3)

**MODEL FORMULATION**

In the system of the fractional order measles model, the total population  is the combination of susceptible , exposed , vaccinated , infected , quarantined  and recovery classes  respectively. At a rate  the susceptible population grows and at a rate the vaccination occurs. At a rate the vaccinated people may lack immunity. The risk of infection for vulnerable people who come into contact with infected people is , with the transmission probability . The transformation from the exposed to the infectious stage occurs at a rate . At a rate  the contagious individuals are separated. and are the infected and quarantined individuals at a rehabilitation rate. In a natural case, the mortality rate is  whereas  and are the case fatality rate (CFR) for infected and quarantined individuals. The fractional order system of the model is

 (4)

with initial conditions,



where are all positive.

**METHODOLOGY**

**Adomian Decomposition Method (ADM)**

Let us consider the system of fractional differential equations,

 (5) where  and . Applying  to (5), we get

 (6)

where 

Using ADM for solving (5), let

 (7)

and  (8)

whereare Adomian polynomials which depend on 

In view of equations (7) and (8), (6) takes the form

 (9)

We set 

 (10)

A parameter is introduced for determining the Adomian polynomials and hence (8) becomes

 (11)

Let , then  (12)

where  (13)

In view of (11) and (12), we get

 (14)

Hence (10) and (14) lead to the following recurrence relations,

 (15)

From (15), we are able to get the approximate solution of (5).

**Laplace Adomian Decomposition Method (LADM)**

Let be a function and the Laplace transformation of be given by

 (16)

The Laplace transform is defined as follows for the fractional derivative of Caputo type.

 (17)

**Differential Transform Method (DTM)**

When  is infinitely continuously differentiable, then  is in Taylor series form as

 (18)

Then the differential transform of the function , denoted by as

 (19)

and the differential inverse transform of the is defined as follows,

 (20)

**ADM methodology**

Consider the fractional order system of equation (4) by using the ADM, we get,

 (21)

and the exact solution of the equation are,

 (22)

Now the infinite series,



and the nonlinear term, .

Hence, the recursive relations are,

 (23)

and the solution can be written as,

 (24)

**LADM methodology**

Applying the Laplace transform to the fractional order system of equation (4), we get,

 (25)

Hence, we get,

 (26)

Hence, the recursive relations are,

 (27)

Hence, the solution can be written as,

 (28)

**DTM methodology**

By using the DTM in the fractional order system of equation (4), we get,

 (29)

**NUMERICAL RESULTS**

We use the numerical values from [24] as,



The general solution of equation (24),

 (30)

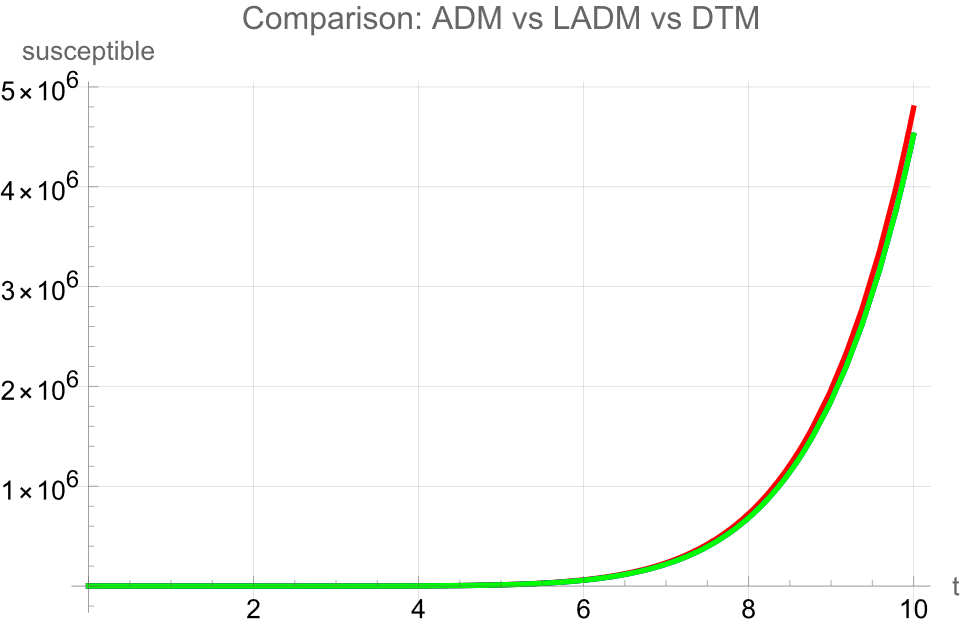
The general solution of equation (28),

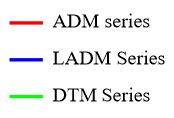
 (31)

The general solution of equation (29),

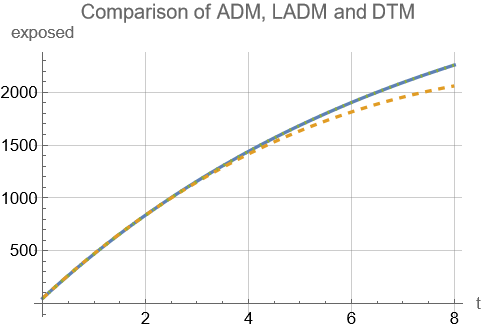
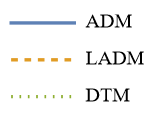
 (32)

**RESULTS and DISCUSSION**

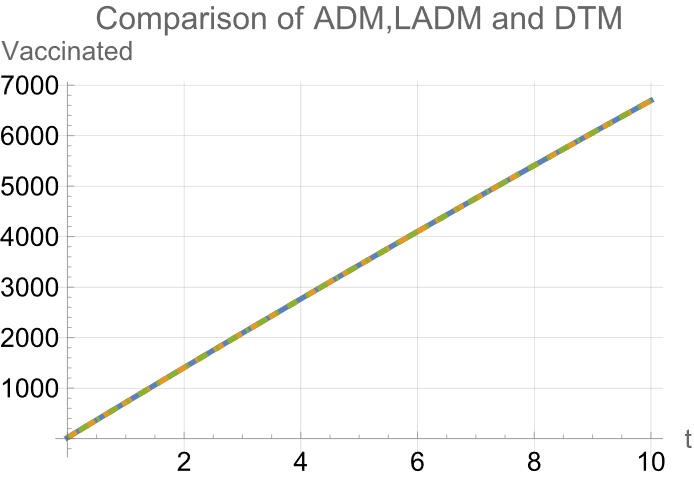


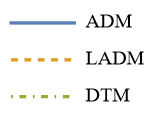


**FIGURE 1.** Rate of change of the Susceptible population

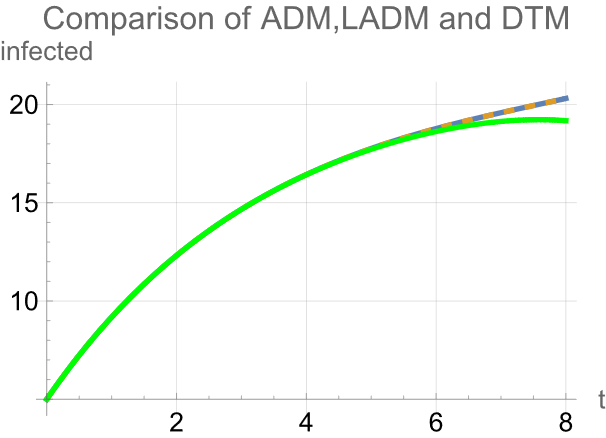


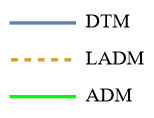
**FIGURE 2.** Rate of change of the exposed population



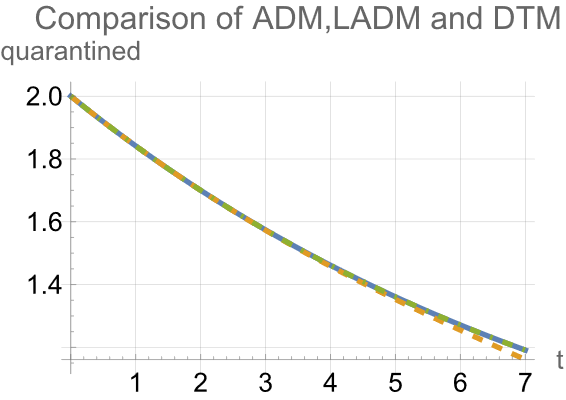


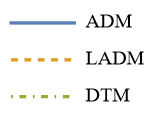
**FIGURE 3.** Rate of change of the vaccinated individuals



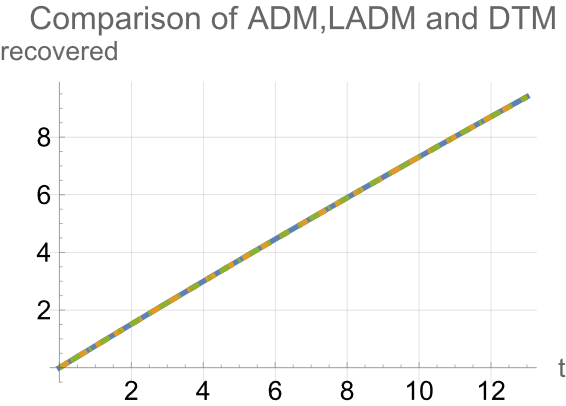


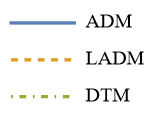
**FIGURE 4.** Rate of change of the infected person





**FIGURE 5.** Rate of change of the quarantined population

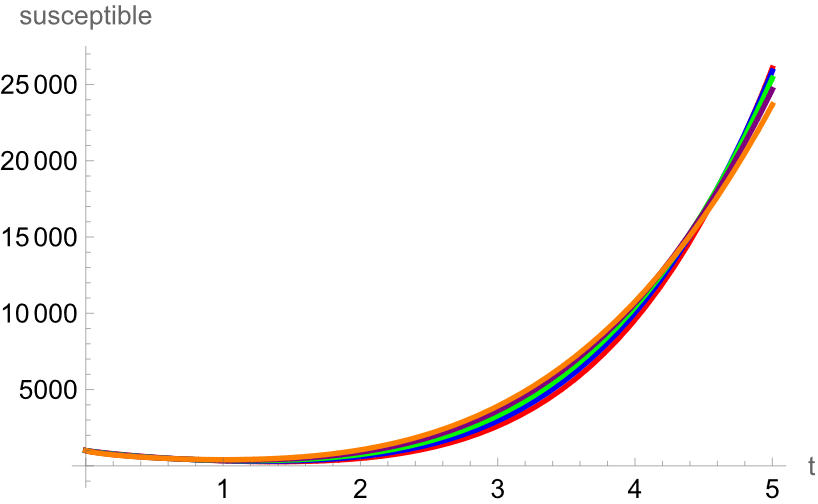
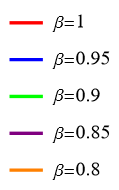




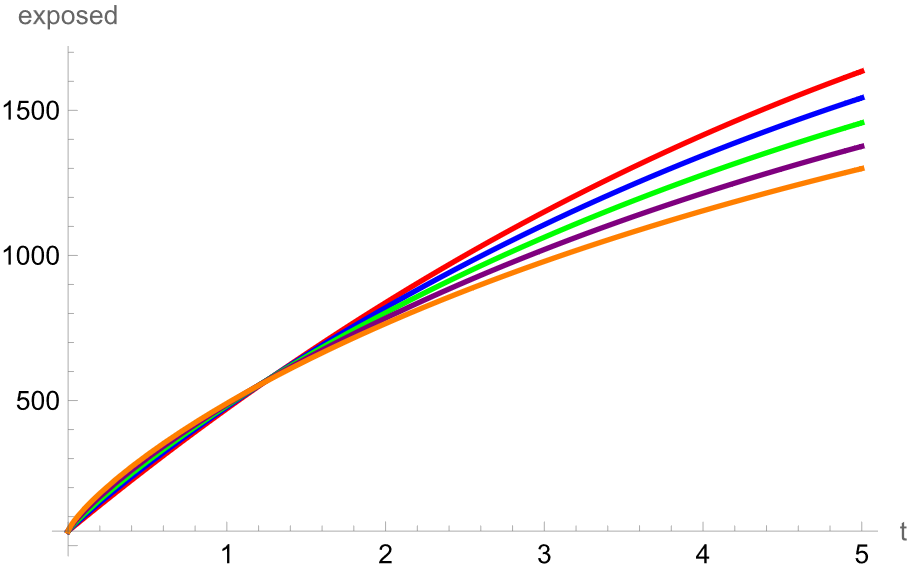
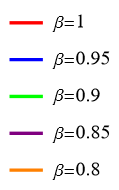
**FIGURE 6.** Rate of change of the recovered individuals

In Fig 1 to 6, we have compared the rate of change of susceptible population, exposed population, vaccinated individuals, infected persons, quarantined population, and recovered individuals, respectively by applying ADM, LADM, and DTM. In the graph  has been considered.

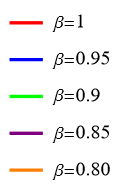
In Fig 7 to 12, the rate of change of susceptible, exposed, vaccinated, infected, quarantined, and recovered populations, respectively has been compared by varying the values of .

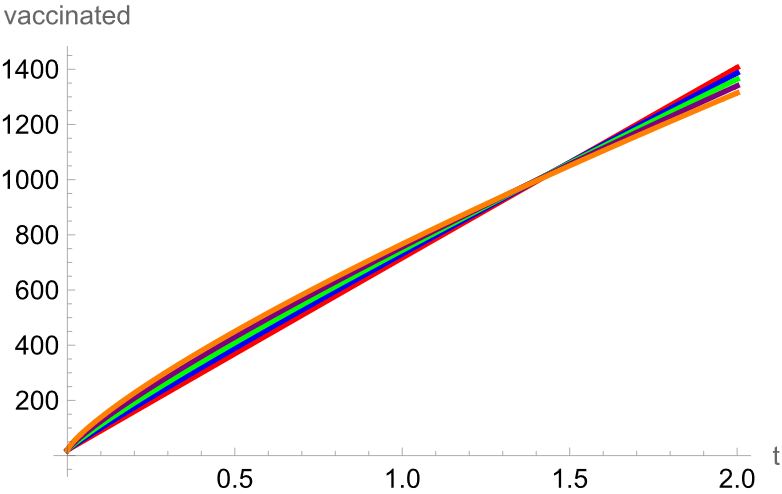


**FIGURE 7.** Susceptible population with varying .

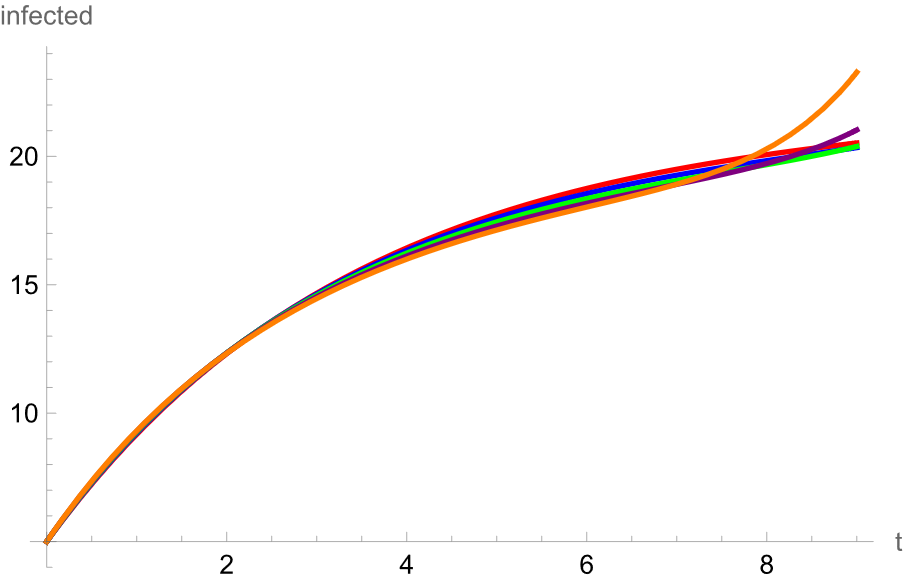


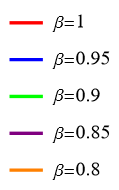
**FIGURE 8.** Exposed population with varying .



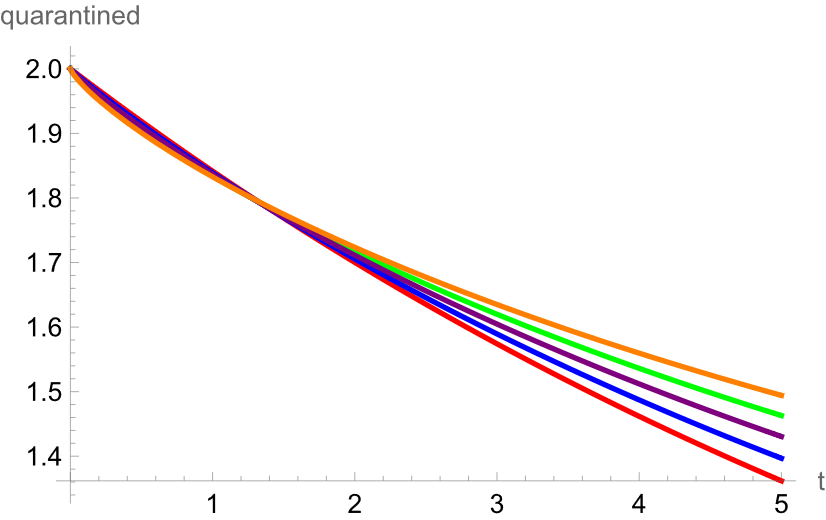
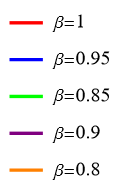


**FIGURE 9.** Vaccinated individuals with varying .

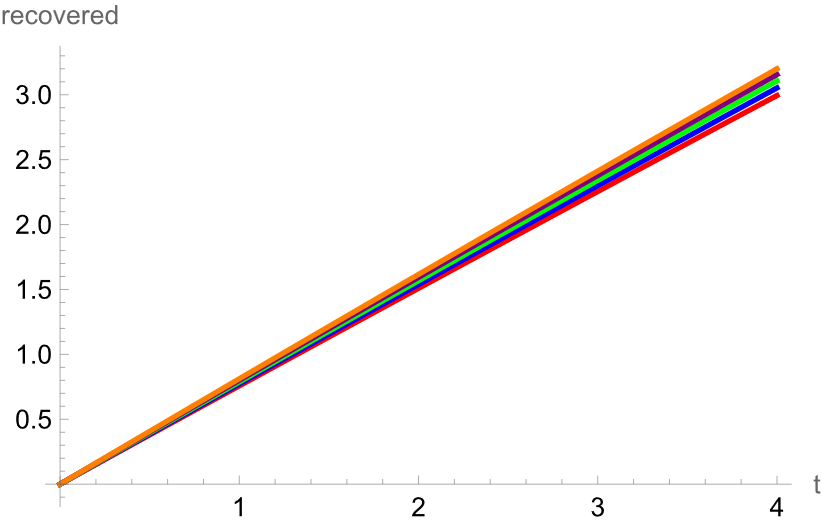


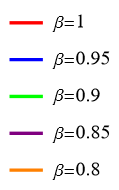


**FIGURE 10.** Infected person with varying .



**FIGURE 11.** Quarantined population with varying .





**FIGURE 12.** Recovered individuals with varying .

In Table 1 to 6, the values for the susceptible, exposed, vaccinated, infected, quarantined, and recovered persons, respectively have been computed, analyzed, and compared by varying the values of time t between 0 to 1and  between 0.8 to 1.

**Table 1**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Susceptible Population** | | | | | |
| **t** |  |  |  |  |  |
| 0 | 1000 | 1000 | 1000 | 1000 | 1000 |
| 0.25 | 749.36 | 731.304 | 713.064 | 694.827 | 676.798 |
| 0.5 | 562.007 | 549.984 | 539.285 | 530.135 | 522.819 |
| 0.75 | 421.311 | 423.323 | 424.647 | 428.726 | 436.172 |
| 1 | 329.58 | 342.228 | 358.511 | 379.299 | 405.673 |

**Table 2**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Exposed Population** | | | | | |
| **t** |  |  |  |  |  |
| 0 | 50 | 50 | 50 | 50 | 50 |
| 0.25 | 160.936 | 170.925 | 181.538 | 192.744 | 204.621 |
| 0.5 | 268.171 | 279.357 | 290.616 | 301.892 | 313.128 |
| 0.75 | 371.767 | 381.121 | 390.066 | 398.552 | 406.533 |
| 1 | 471.785 | 477.525 | 482.527 | 486.771 | 490.246 |

**Table 3**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Vaccinated Individuals** | | | | | |
| t |  |  |  |  |  |
| 0 | 20 | 20 | 20 | 20 | 20 |
| 0.25 | 194.745 | 211.099 | 228.63 | 247.373 | 267.353 |
| 0.5 | 369.075 | 388.709 | 408.791 | 429.253 | 450.016 |
| 0.75 | 542.992 | 561.294 | 579.295 | 596.891 | 613.974 |
| 1 | 716.496 | 730.554 | 743.68 | 755.775 | 766.737 |

**Table 4**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Infected Person** | | | | | |
| **t** |  |  |  |  |  |
| 0 | 5 | 5 | 5 | 5 | 5 |
| 0.25 | 6.16069 | 6.18151 | 6.2006 | 6.21784 | 6.23306 |
| 0.5 | 7.24084 | 7.27548 | 7.3065 | 7.3337 | 7.35686 |
| 0.75 | 8.24604 | 8.28856 | 8.32554 | 8.35676 | 8.38201 |
| 1 | 9.18148 | 9.22688 | 9.26487 | 9.2953 | 9.31806 |

**Table 5**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Quarantined Population** | | | | | |
| **t** |  |  |  |  |  |
| 0 | 2 | 2 | 2 | 2 | 2 |
| 0.25 | 1.95869 | 1.95491 | 1.95088 | 1.9466 | 1.94207 |
| 0.5 | 1.91854 | 1.9142 | 1.9098 | 1.90537 | 1.90091 |
| 0.75 | 1.87951 | 1.87572 | 1.87203 | 1.86849 | 1.86511 |
| 1 | 1.84159 | 1.83897 | 1.83661 | 1.83451 | 1.8327 |

**Table 6**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Recovered Individuals** | | | | | |
| **t** |  |  |  |  |  |
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0.25 | 0.18989 | 0.193775 | 0.19741 | 0.200768 | 0.203819 |
| 0.5 | 0.37941 | 0.387145 | 0.39438 | 0.401057 | 0.407121 |
| 0.75 | 0.56856 | 0.580112 | 0.590909 | 0.600868 | 0.609907 |
| 1 | 0.757343 | 0.772675 | 0.786999 | 0.800204 | 0.812179 |

**CONCLUSION**

In this work, we have studied the fractional order measles epidemic model mathematically by applying the ADM, DTM, and LADM, which are semi-analytical methods, and compared the results both numerically and graphically.

In the future, we propose to study several other real-life fractional and functional models by applying different analytical, semi-analytical, and numerical methods.

**REFERENCES**

1. I. Podlubny, Fractional Differential Equations, ACADEMIC PRESS, San Diego, 198,1998.
2. E. U. Agom and A. M. Badmus, International Journal of Engineering Science Invention, 4(11), 60-65, (2015).
3. N. T. Shawagfeh, (1999). *J. Frac. Calc*, *16*, 27-33, (1999).
4. S. Momani and N. Shawagfeh, *Applied Mathematics and Computation*, *182*(2), 1083-1092, (2006).
5. K. Al-Khaled and S. Momani, *Applied Mathematics and Computation*, *165*(2), 473-483, (2005).
6. H. Jafari and V. Daftardar-Gejji *Journal of Computational and Applied Mathematics*, *196*(2), 644-651, (2006).
7. M. A. Hussein, *Scientific Research Journal of Multidisciplinary*, *2*, 1-10, (2022).
8. M. Y. Ongun, *Mathematical and Computer Modelling*, *53*(5-6), 597-603, (2011).
9. M. Farman, M. U. Saleem, A. Ahmad and M. O. Ahmad, *Ain Shams Engineering Journal*, *9*(4), 3391-3397, (2018).
10. M. S. Budiman and D. Salim, *Hilbert Journal of Mathematical Analysis*, *2*(1), 026-033, (2023).
11. M. Alsulami, M. Al-Mazmumy, M.A. Alyami and A. S. Alsulami, *Mathematics*, *12*(22), 3499, (2024).
12. J. M. W. Munganga, J. N. Mwambakana, R. Maritz, T. A. Batubenge and G. M. Moremedi, *International Journal of Mathematical Education in Science and Technology*, *45*(5), 781-794, (2014).
13. F. Mirzaee, *Applied Mathematical Sciences*, *5*(70), 3465-3472, (2011).
14. N. Patil and A. Khambayat, *Research Journal of Mathematical and Statistical Sciences \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ISSN*, *2320*, 6047, (2014).
15. I. A. H. Hassan *Applied Mathematical Modelling*, *32*(12), 2552-2559, (2008).
16. R. K. Saeed and B. M. Rahman, *Aust. J. Basic Appl. Sci*, *5*(4), 201-206, (2011).
17. B. Liu, X. Zhou and Q. Du, *Applied Mathematics*, *6*(3), 585, (2015).
18. B. Ibis, M. Bayram and A. G. Agargun, *European Journal of Pure and Applied Mathematics*, *4*(2), 129-141, (2011).
19. R. Almeida, N. R. Bastos and M. T. T. Monteiro, *Mathematical Methods in the Applied Sciences*, *39*(16), 4846-4855, (2016).
20. K. S. Nisar, A. Ahmad, M. Inc, M. Farman, H. Rezazadeh, L. Akinyemi and M. M. Akram, AIMS Mathematics,7(5),8408-8429, (2022).
21. S. Nana-Kyere, B. Seidu and K. Nantomah, *Journal of Applied Mathematics*, *2024*(1), 5533885, (2024).
22. B. Guo, A. Khan and A. Din, *Fractal and Fractional*, *7*(2), 130, (2023).
23. H. T. Alemneh and A. M. Belay, *Discrete Dynamics in Nature and Society*, *2023*(1), 9353540, (2023).
24. P. N. Akuka, B. Seidu, B. Okyere and S. Abagna, *Scientifica*, *2024*(1), 899730, (2024).