**A Fuzzy Approach to the Dynamics of Tumor Growth in Uncertain Environments**

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**Abstract**: Predicting the behaviour of growth processes in biology and engineering is challenging when key parameters are uncertain. In the logistic model, for example, the growth rate is rarely known with precision, especially in clinical or experimental settings. Here, we adapt the classical logistic framework by expressing the growth rate as an interval type-2 trapezoidal fuzzy number. This choice allows the model to reflect not only imprecise data but also uncertainty about the degree of imprecision itself. While the idea is illustrated through tumour growth simulations, the method is not limited to cancer research it can be applied to many systems where the growth rate fluctuates or is difficult to estimate. The results suggest that the fuzzy extension produces predictions that are more cautious in early growth stages and more realistic when data are incomplete, without losing the interpretability of the standard logistic curve.

***Keywords:*** *Tumor Growth, Logistic Model, Interval type-2 trapezoidal fuzzy set* ***(****IT2TrFS)*

**INTRODUCTION**

Tumor growth is one of the central subjects of cancer research. This is based on its intricate biological nature and its role in cancer diagnosis, therapy, and prognostication. Tumors arise when normal cells develop genetic and epigenetic alterations that disturb systems involved in cell growth and programmed cell death. Such disturbance is a cause of uncontrolled and abnormal cell growth, leading to a mass that may be benign or malignant. The growth of a tumor is affected by numerous internal and external factors. Some of these factors include the immune response, blood supply, and surrounding environment. All these contribute to the challenges in predicting and controlling the behaviour of a tumor in clinical conditions.

To better understand and predict tumor progression, a variety of mathematical models have been proposed over the years. Among these, ordinary differential equation (ODE) models such as the logistic, Gompertz, Bertalanffy, exponential and linear models **(Koziol & Falls et al.2020)**. Logistic and Gompertz model describe sigmoidal growth behaviour observed in real tumor data. The logistic model, in particular, stands out for its simplicity and interpretability **(Simpson & Browning et al. 2022)**. It assumes that tumor cells grow rapidly at first when resources are abundant, and then slow down as the tumor size approaches a maximum limit, or carrying capacity, due to environmental constraints like nutrient depletion and immune response. Several studies have shown that logistic-type models not only provide biologically plausible results but also align well with empirical tumor growth data **(Zabor & Reddy et al. 2022)**.

However, the problem with applying the logistic model to actual medical data is that the growth rate is not fixed. In practice, the rate of growth of tumor cell varies person to person and several environmental factors **(Wang, & Zhan et al. 2022)**. For the above-mentioned factors, it hard to apply the logistic model in a precise way without addressing this uncertainty. A fixed growth rate may not accurate for the said model, especially when patient conditions vary or when data is incomplete **(Boadh & Grover et al. 2022)**.

In order to overcome this shortcoming, this paper employs Interval Type-2 Fuzzy Numbers (IT2FNs) to express the uncertain growth rate of the logistic model. Fuzzy logic is a powerful mathematical tool for handling vague or imprecise data, which commonly occurs in medical related data. The traditional (type-1) fuzzy number have a fixed membership function, but type-2 fuzzy number admits uncertainty even in the membership values. The Interval type-2 fuzzy sets are a special case, where the uncertainty is modelled as an interval, simplifying the computation without sacrificing the imprecision.

By using IT2FNs for the growth rate *r*, this work is intended to create a more general and more precise tumor growth model. Rather than giving *r* a single value, we are using it as an interval representing potential variation, so the model can accurately mirror actual biological conditions. This integration of the logistic growth model and fuzzy logic provides a realistic and adaptive model of tumor growth, particularly in situations where the information is uncertain or varies from patient to patient.

**MATHEMATICAL LOGISTIC MODEL**

The logistic growth model is one of the most commonly used mathematical models for describing the tumor growth rate of biological systems. The model is particularly useful because it captures both the primary rapid proliferation of tumor cell and then eventual slowdown as resources such as nutrients, oxygen and space become limited. The logistic model based on the idea that the tumor growth is proportional not only to the current size of the tumor but also to the remaining capacity of the environment to support further growth.

Mathematically, it is expressed as:

 (1)

Here,  represents the size of tumor with respect to time *t*,  is the growth rate, and  is the carrying capacity, which indicates the maximum size of the tumor in human body. Primarily, when the tumor size is small, the term is close to 1, so the growth is nearly exponential. When the tumor size grows and approaches to the carrying capacity K, the growth rate slowdown. This approach from a S-shaped (sigmoidal) behaviour, so this makes the logistic model more realistic. In the context of oncology, the logistic model provides a simple yet effective framework to study how tumors progress over time and to estimate key biological features like tumor doubling time and treatment impact. However, despite its usefulness, the logistic model assumes fixed parameters, which can limit its accuracy in representing the highly variable and uncertain nature of real tumor growth in different patients and conditions.

**Solution of the Logistic Model:**

The logistic Model was solved using the “dsolve” command in MATLAB, to understand the behaviour of the logistic model and how the tumor size changes over the time. By using the MATLAB software and simulating the differential equation the tumor growth was plotted.



**Figure-1**

Fig - 1shows the solution of the logistic Here, we take , , and units,  represent the initial size of the tumor.

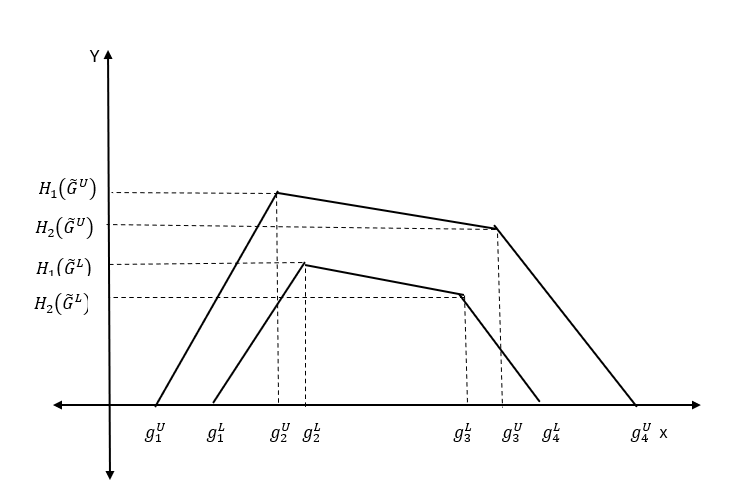
model clearly a sigmoid-shaped curve, which is typical behaviour for the logistic model. In this graph the rate of growth, which is considered constant, grows slowly, then it enters to the phase where the tumor growth is rapid, but after some times, growth rate is starts decreasing due to the reach of the limit. This happens because, the growth rate of tumor is bigger, the cells in the inner region stop getting enough oxygen and nutrients. Due to this limitation the growth rate slows down and the tumor size is approaching to the maximum limit, which is known as the carrying capacity (K). So, the figure looks like an S-shaped(sigmoid), where the growth starts fast and then flattens out as it reaches the limit. This simulation helps us to visualize and understand how tumor growth is not infinite but instead self-limiting due to biological constraints like lack of space, oxygen, and nutrients. The constant rate of growth considered here is not a realistic approach to the tumor growth as the growth rate is not constant in real life scenarios, so by considering a fuzzy growth rate we proceed in the next part of the paper.

**INTERVAL TYPE -2 TRAPEZOIDAL FUZZY SET (IT2TRFN):**

In such real-world biological systems as the growth of a tumour, parameters such as the growth rate  often cannot be calculated accurately because of factors such as variation between patients, variation in the tumour micro environment, poorly calibrated clinical measurements, and incomplete biological knowledge. Mechanical models of traditional crisp threshold-to-threshold mapping cannot accommodate this natural fuzziness when explaining the mutation or can result in over-simplified or non-robust predictions. To handle this type of uncertainty more effectively, we represent the growth rate as an Interval Type-2 Trapezoidal Fuzzy Set (IT2TrFS) . This approach extends the conventional (Type-1) fuzzy set theory by introducing secondary membership functions, allowing the model to handle second-order uncertainties i.e., uncertainty about the membership grades themselves.

An IT2TrFS is characterized by a Footprint of Uncertainty (FOU), which is the area bounded by:

* An Upper Membership Function (UMF) and
* A Lower Membership Function (LMF) 

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**Figure-2**

**IT2TrFN (Interval type -2 trapezoidal fuzzy number)**

A fuzzy set where each element has a membership value represented as an interval, and the shape of the membership function is trapezoidal is a IT2TrFS. An arbitrary IT2TrFS may require complex modelling of secondary membership functions, whereas an IT2TrFN simplifies it using a standard shape. An IT2TrFN is essentially an IT2TrFS defined on ℝ with a trapezoidal membership function.

Considering  as an Interval type-2 trapezoidal fuzzy number (IT2TrFN) as , where , i = 1, 2 3,  ,  and, be the IT2TrF number.  and  are defined by lower membership function  and upper membership function  as follows:



and



**Alpha cut of IT2TrFN**

If is an IT2TrFN, then alpha cut of , where , is defined as follows:



where  , 

, 

**Interval approximation of IT2TrFN:** The interval approximation of IT2TrFN  is one of the methods for defuzzification and is defined as follows:



The mean, also known as the de-fuzzified scalar value, is a non-fuzzy value that denotes the center or expected value of an Interval Type-2 Trapezoidal Fuzzy Number (IT2 TrFN).

Mean (de-fuzzified to a scalar) value of the interval approximation of IT2TrFN  is defined as follow:

 (2)

If we take  , then above Equation (2)

. Again, if we take  , then i.e., if no fuzziness occurs, the de-fuzzified value of IT2TrFS correlates to the crisp number.

**MATHEMATICAL MODELLING OF LOGISTIC MODEL USING IT2TRFS GROWTH RATE**

To address the uncertainty associated with tumor progression parameters, we generalized the traditional logistic model within a fuzzy logic-based framework. In this extended model, the growth rate  is not considered as a precise constant but instead represented as an interval type-2 trapezoidal fuzzy number. The imprecision and uncertainty frequently arise in biological system, especially when the data of patient is vague or incomplete. By introducing the fuzzy growth rate and adding the type-2 (Trapezoidal) fuzziness, the logistic model is made more adaptable and realistic by reflecting that the growth rate may vary due to environmental, genetics, or treatment-related factors. This model offers a more reliable depiction of tumor growth, particularly in situations that are ambiguous.

Mathematically, the fuzzy growth model is expressed as:

 (3)

Here,  represents the population of the infected tumor cell, is the fuzzy rate of the growth and  is representing the carrying capacity.

**Solution of the Fuzzy Logistic Model**

To study tumor growth rate under uncertainty, we solved the fuzzy growth rate of logistic model where the growth rate  taken as a type-2 trapezoidal fuzzy number. This adds a degree of flexibility to the model by allowing it to account for ambiguity in biological data, especially in estimating how fast the tumor grows. Since the model involves fuzzy parameters, we applied the α-cut method to handle the fuzziness and reduce the fuzzy differential equation to a set of interval-based crisp equations. For different α-levels (e.g., from 0 to 1), we generated the corresponding interval bounds of the solution. Then, we used MATLAB to solve these interval-based equations using the ode45 solver for each α-cut. After defuzzification, we obtained a curve showing the possible range of tumor growth over time.



**Figure-3**

Here, we consider ,  with initial tumor size units. This graph shows that a single curve, representing the uncertainty of the tumor size prediction. Like the classical case, the shape of each curve in the fuzzy band is sigmoid, reflecting the natural progression of the tumor growth rate. This graph shows that the time is increases, the tumor size grows rapidly at fast but gradually slows down and approach to the carrying capacity due to the biological limitation like restrict oxygen and nutrients supply. This fuzzy solution helps to understand how tumor growth may be behaved in real – life cases, where exact parameters are not always known with precision.

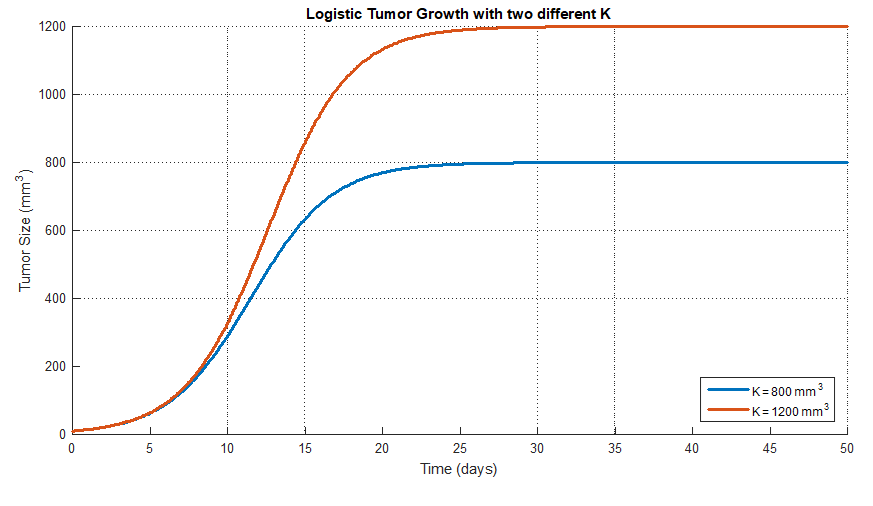
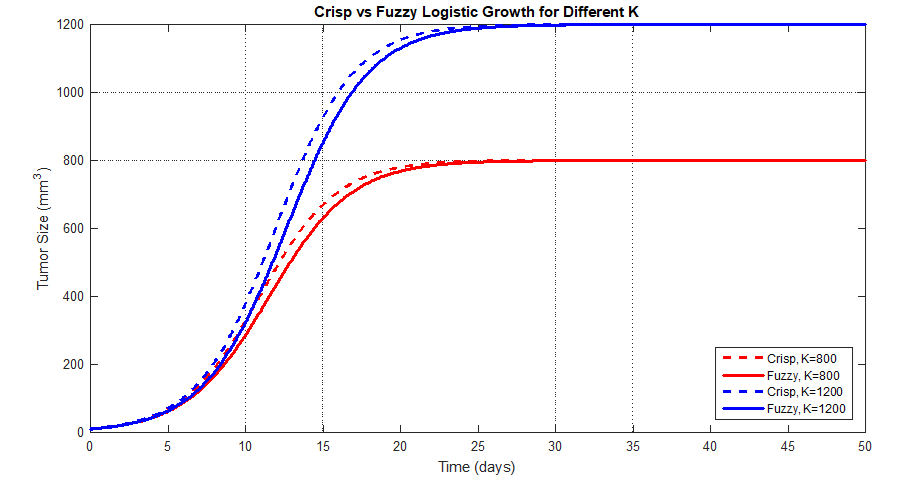


**Figure-4**

Fig - 4 represents a comparison between the classical crisp model and the fuzzy logistic model after de-fuzzification by considering the , K = 1000. As shown in the graph both the models exhibit the expected S-shaped curve, indicating typical tumor growth behaviour starting slow, accelerating rapidly, and then tapering off as it approaches the carrying capacity. In the graph the red dashed line represents the crisp value of the logistic model. Primarily, the growth patterns are quite similar, but as time progresses, a noticeable difference emerges.

**RESULT AND DISCUSSION**

If we consider two different carrying capacities, as in Fig – 5, the graph shows that even with r being moderate, a small K will decrease peak growth; yet with large K, both an early inflection and an extremely large peak growth are produced.

**Figure 5 Figure 6**

In Fig – 6, it can be observed the different carrying capacities being considered with fuzzy rate of growth, shows the same kind of result as in fig -5, i.e; though a reduced plateau does not imply eradication, decreased K can increase the window of time before the tumor reaches clinically dangerous sizes for other treatments to be effective. An invasive adaptable tumor cell may be selected for by a lower K. Tumor size can be controlled without actually increasing cell death via treatments that effectively reduce the tumor's carrying capacity (such as anti-angiogenic medicines that limit the vasculature and nutrition supply). Practically, lowering  
K is sometimes a therapeutic target. Anti-angiogenic therapy, for example, limits the development of blood vessels, reducing nutrient supply and, in turn, reducing K. This does not kill tumor cells directly but prevents them from exceeding a smaller sustainable size (Fukumura, et, al; 2018).

The fuzzy logistic model reflects a slightly more conservative growth path, due to the uncertainty in the growth rate parameter being modelled as a type-2 trapezoidal fuzzy number. This fuzziness captures the natural variability in tumor behaviour more realistically than the crisp model. Eventually, both curves level off near the carrying capacity, confirming that resource limitations like oxygen and nutrients ultimately restrict tumor expansion. This comparison figure shows that the fuzzy growth model’s strength in providing a range-aware prediction that is better suited for real clinical settings where exact growth rates are difficult to determine.

**Table -1**

|  |  |  |
| --- | --- | --- |
| **Growth Rate ()** | **Tumor size in mm3**  **Crisp value** | **Tumor size in mm3**  **Fuzzy value** |
| 0.2 | 995.5255 | 992.6442 |
| 0.25 | 999.6312 | 999.3112 |
| 0.3 | 999.9697 | 999.9356 |
| 0.35 | 999.9975 | 999.1140 |
| 0.4 | 999.998 | 999.9994 |

The comparison table as well as Fig 8 clearly indicate that tumor size goes on increasing as the growth rate increases, in both crisp logistic model and type-2 fuzzy logistic model.



**Figure-7**

As expected, with higher growth rate the tumor grows faster and it reach the carrying capacity earlier. But in case of fuzzy model, where a type-2 trapezoidal fuzzy number is used for growth rate, it introduces some level of uncertainty which is closer to what happen in real biological condition. For the same value of growth rate, the fuzzy model generally shows slightly smaller tumor size at earlier time steps compared to crisp one. This slower growth in beginning happens because fuzzy model considers a range of possible growth rate instead of just one, so the prediction is more cautious. As growth rate become higher, both models show increased tumor size but the fuzzy model gives more spread in output which look more realistic. Overall, this comparison shows the benefit of using fuzzy modelling specially when we deal with uncertain or patient-specific data. It helps to capture the kind of variability that real tumor growth usually has, which the crisp model often misses.



**Figure-8**

Fig -7 shows that how changing the growth rate affects the final tumor size for both the crisp logistic model and the fuzzy logistic model that uses a type-2 trapezoidal fuzzy number for the growth rate. As the growth rate goes up from 0.2 to 0.4, the tumor size at the final time step also increases in both models. But there's a noticeable difference between the two. The crisp model (solid black line with circles) rises steeply at first and then flattens out close to the carrying capacity. The fuzzy model (dashed red line with squares), on the other hand, grows a bit more cautiously, especially when  is low. That’s because of the uncertainty built into the fuzzy growth rate,which makes the predictions a bit broader and less aggressive. As increases, both models start to come together and reach the same upper limit which is basically the carrying capacity or biological maximum. This figure really highlights why fuzzy modelling is important in medical field, since it helps account for uncertainty in real data and doesn’t give too optimistic predictions too early on. In the above figures, it clearly shows, if the growth rate increases, then the tumor size also increases and when it comes closer to the carrying capacity, the growth rate slows down may be due to the lack of oxygen, nutrition, and space.

**CONCLUSION**

This work aimed to study tumor growth behaviour using both the classical logistic model with type-2 fuzzy extension that includes uncertainty in the growth rate. While the logistic model does capture the overall sigmoidal growth pattern quite well, it assumes everything is known precisely, which is hardly ever the case in real biological or clinical data. So, by applying a type-2 interval trapezoidal fuzzy number to the growth rate, we tried to reflect more realistic conditions where parameters are not always fixed or exact. The simulation results showed that the fuzzy model tends to grow a bit more slowly at the start compared to the crisp one. This isn’t surprising, since it’s accounting for uncertainty and doesn’t just assume one fixed growth value. As the growth rate increased, both models eventually approached the same carrying capacity, but the fuzzy model gave more cautious predictions, especially in the early stages. These results suggest that fuzzy modelling could be more suitable in clinical cases where patient data is incomplete or uncertain. It might not give the fastest result, but maybe a safer one. In future, more work is needed to include treatment response or individual patient differences, which could make the model even more practical.

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