Solving the American Option Pricing Problem under Uncertainty with Fuzzy ADM

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**Abstract.** In financial markets, American options offer the flexibility of early exercise, making their valuation more complex than European options. Traditional pricing models often assume crisp input parameters such as volatility, interest rates, and asset prices. However, in real-world scenarios, these parameters are often imprecise or uncertain due to market volatility, incomplete informa- tion, or investor sentiment. To better reflect this inherent uncertainty, this study introduces a fuzzy framework for the numerical valuation of American options using the Adomian Decomposition Method (ADM). By incorporating fuzzy numbers—specifically triangular fuzzy numbers—into the classical Black-Scholes equation, and handling the early exercise feature as a free boundary problem, we develop a robust fuzzy ADM-based model. The proposed method yields upper and lower bounds of the American option price as fuzzy membership functions. Comparative analysis with crisp ADM and other numerical methods demonstrates the efficiency and accuracy of the fuzzy ADM in capturing the impact of uncertainty on option pricing. The findings provide deeper insights into the behavior of American options under uncertain market conditions and offer a valuable decision-support tool for investors and risk managers.

**Keywords:** European option pricing; Fuzzy uncertainty; Triangular fuzzy numbers; Black–Scholes model; Adomian Decom- position Method (ADM); Fuzzy partial differential equations; *α*-cut representation; Numerical approximation; Fuzzy financial modeling.

# INTRODUCTION

The pricing of financial derivatives, particularly American options, remains a critical area of study in quantitative finance. Unlike European options, American options grant the holder the right to exercise the option at any time before or at maturity. This early exercise feature introduces additional complexity into their valuation, necessitating the solution of free boundary problems. Traditional methods such as the binomial tree, finite difference methods, and least-squares Monte Carlo have been extensively used to approximate American option prices under the assumption that market parameters such as volatility, interest rates, and asset prices are precisely known.

However, in practical financial markets, these parameters are often subject to imprecision and vagueness due to rapidly changing economic conditions, incomplete data, and investor behavior. Conventional stochastic models may not sufficiently capture this type of uncertainty, which is more appropriately modeled using fuzzy set theory. Intro- duced by Zadeh, fuzzy theory allows parameters to be represented as fuzzy numbers, reflecting uncertainty not as randomness but as ambiguity or lack of clarity in the available information.

The integration of fuzzy logic into option pricing models has received increasing attention in recent years. Fuzzy extensions of the Black–Scholes model have been explored for European options, demonstrating that incorporating uncertainty through fuzzy sets provides a more realistic range of prices, especially under volatile or incomplete market information. However, limited work has been done on extending these ideas to American options, where the early exercise feature introduces further challenges.

In this context, the Adomian Decomposition Method (ADM) emerges as a powerful analytical tool for solving nonlinear and complex differential equations without linearization or discretization. When adapted to fuzzy environ- ments, the fuzzy Adomian Decomposition Method (FADM) can be used to solve fuzzy differential equations arising in the valuation of American options. This approach provides an efficient and flexible method to derive approximate solutions and capture the influence of parameter uncertainty.

This study aims to develop a fuzzy ADM-based framework for the numerical valuation of American options. We model the American option pricing problem using a fuzzy Black–Scholes partial differential equation, where key parameters are represented as triangular fuzzy numbers. The fuzzy ADM is then applied to solve the resulting free boundary problem, yielding the option price as a fuzzy membership function. Numerical simulations are conducted to analyze the behavior of the fuzzy option price and the effect of parameter uncertainty on the early exercise boundary.

# PRELIMINARIES

In this section, we briefly recall the basic concepts and mathematical tools necessary for developing a fuzzy framework for American option pricing using the Adomian Decomposition Method (ADM). This includes a review of fuzzy numbers, operations on fuzzy functions, and an overview of the ADM in both crisp and fuzzy settings.

## Fuzzy Numbers and *α*-cuts

A *fuzzy number* is a convex, normalized fuzzy set on the real line with a piecewise continuous membership function . Among various types of fuzzy numbers, *triangular fuzzy numbers (TFNs)* are frequently used due to their simplicity and interpretability.

A triangular fuzzy number is characterized by a triplet, where : left endpoint (lower bound), : mode (most likely value), : right endpoint (upper bound). The membership function of a TFN is defined as:

An equivalent representation of a fuzzy number is via *-cuts*. The -cut of a fuzzy number for is the closed interval:

where:

These intervals facilitate interval arithmetic for fuzzy-valued functions.

## Fuzzy-Valued Functions

Let be a fuzzy-valued function defined on a real interval . Using -cuts, the fuzzy function can be represented as:

Differentiability and integration of fuzzy-valued functions are often defined in terms of the differentiability of their -cuts or using the *Hukuhara derivative*, which generalizes classical derivatives to fuzzy-valued functions. In this work, we use the extension principle and -cut method to manage uncertainty.

## Adomian Decomposition Method (ADM)

The ADM is a semi-analytical technique used to solve linear and nonlinear differential equations without requiring discretization or linearization. Consider a differential equation of the form:

where is the highest-order linear differential operator, is the remaining linear part, is the nonlinear operator, is the source term or known function. Applying the inverse of , the equation becomes:

The solution is decomposed as a series:

and the nonlinear term is decomposed using *Adomian polynomials*:

where are constructed from the components .

The recursive relations for are obtained iteratively:

## Fuzzy ADM (FADM)

In a fuzzy setting, ADM is applied to the -cut representation of the fuzzy differential equation. The fuzzy solution is constructed as an interval solution for each -level:

This approach preserves the uncertainty structure throughout the computation and provides the final solution as a fuzzy set (typically visualized through membership functions).

# MODEL FORMULATION

In this section, we present the mathematical formulation of the fuzzy American option pricing model. The standard Black–Scholes partial differential equation (PDE) is modified to incorporate fuzzy uncertainty in key parameters, such as the volatility, risk-free interest rate, and asset price. Additionally, the early exercise feature of American options introduces a free boundary problem, which is addressed within the fuzzy framework.

The classical Black–Scholes PDE for the price of an American option is given by:

with the constraint:

where the equality holds when early exercise is optimal. The associated boundary and terminal conditions depend on whether it is a call or put option. Here, denotes the asset price, the time to maturity, the strike price, the risk-free interest rate, and the volatility of the asset. In real-world markets, the parameters , , and are often subject to uncertainty. We model this vagueness using triangular fuzzy numbers:

Substituting the fuzzy parameters into the Black–Scholes PDE leads to the fuzzy differential equation:

with fuzzy terminal and boundary conditions:

To solve the fuzzy PDE using the Adomian Decomposition Method, we convert the fuzzy equation into a system of interval-valued equations using the -cut representation. For each , the fuzzy parameters are represented as intervals

Consequently, the fuzzy option price is represented as:

where and satisfy two separate PDEs corresponding to the left and right bounds of the solution at each -level.

For American options, the optimal exercise boundary is fuzzy and must be determined as part of the solution. Let denote the fuzzy early exercise boundary. The fuzzy value function satisfies:

for the fuzzy call option. These conditions must be solved simultaneously with the fuzzy PDE.

The goal is to solve the above fuzzy system using the fuzzy Adomian Decomposition Method (FADM) to obtain the fuzzy option price and the fuzzy early exercise boundary . The results will be presented in the form of membership functions for a range of -levels.

# THEORETICAL RESULTS

In this section, we present some theoretical results related to the American option pricing problem under fuzzy uncertainty using the Adomian Decomposition Method (ADM).

**Theorem 3.1 (Existence of Fuzzy Solution)** *Let be triangular fuzzy numbers with compact -cuts. Then the fuzzy American option pricing problem*

with appropriate terminal and early exercise conditions, admits at least one fuzzy solution represented by its -cuts.

*Proof.* For each , the fuzzy parameters reduce to closed intervals, e.g.

Thus, the fuzzy PDE reduces to a family of interval-valued PDEs parameterized by . Classical PDE theory guarantees the existence of a solution for each . By assembling all -cut solutions, we obtain a fuzzy solution . Hence, existence is established.

**Theorem 1 (Convergence of FADM)** *Suppose the nonlinear fuzzy operator in the decomposition is Lipschitz continuous on the solution domain. Then the series solution obtained by the Fuzzy Adomian Decomposition Method*

converges uniformly to the exact fuzzy solution .

*Proof.* By Lipschitz continuity, we have

for some . This ensures that the decomposition series forms a Cauchy sequence in the fuzzy Banach space of continuous fuzzy-valued functions. Therefore, the FADM solution converges uniformly to the exact fuzzy solution.

**Theorem 2 (Bounding Property)** *The crisp ADM solution obtained using deterministic parameters lies inside the fuzzy band of the FADM solution:*

*Proof.* When , the fuzzy parameters collapse to their crisp midpoints, which correspond to the deterministic model. Hence, the crisp ADM solution coincides with the core of the fuzzy solution. For , the fuzzy parameters enlarge the uncertainty band, and thus the crisp solution always lies within the fuzzy envelope.

## Numerical Method and Implementation Using FADM

In this section, we present the numerical procedure for solving the fuzzy Black–Scholes equation for American options using the Fuzzy Adomian Decomposition Method (FADM). The approach involves decomposing the fuzzy partial differential equation into interval-valued subproblems at each -cut level and applying the Adomian decomposition to derive approximate solutions iteratively

Let the fuzzy solution be represented by its -cut:

We solve the fuzzy Black–Scholes PDE at each level as two separate deterministic PDEs for the lower and upper bounds:

subject to corresponding boundary and terminal conditions.

Using ADM, we express the solution as a series:

and decompose the nonlinear terms using Adomian polynomials:

The recursive scheme is given by:

where is the inverse operator with respect to time or space, applied as appropriate.

Since American options involve an optimal exercise boundary , we define a fuzzy boundary with:

The fuzzy value matching and smooth pasting conditions at each level are:

These are solved iteratively together with the FADM expansion, using suitable boundary approximations or interpolation schemes

# RESULTS AND DISCUSSION

In this section, we present and analyze the numerical results obtained using the proposed fuzzy Adomian Decomposition Method (FADM) for pricing American options. The outcomes are compared with classical (crisp) models to highlight the impact of fuzzy uncertainty in key parameters such as volatility, interest rate, and asset price. The membership functions of the fuzzy option prices are visualized, and the behavior of the fuzzy early exercise boundary is discussed.

We consider a triangular fuzzy representation for the parameters involved in the option pricing model. The parameters are chosen as follows

Fuzzy volatility: , Fuzzy interest rate: , Fuzzy initial stock price: , Strike price: , Maturity: year.

The computational domain is discretized using uniform grids, and the -cuts are sampled at increments of 0.1 over the interval .

The fuzzy solution obtained through FADM is represented using its -cuts. The resulting membership functions show the range of possible option prices for different levels of confidence (-levels). Figure 1 illustrates the fuzzy price of an American call option at as a function of the stock price . Figure 2 presents the fuzzy early exercise boundary for various -levels. As decreases, the uncertainty in the boundary widens, indicating that early exercise decisions are also affected by the imprecision in model parameters.

Fuzzy Membership Function of American Call Option Price (FADM)

Alpha levels

= 0.0

= 0.2

= 0.4

= 0.6

= 0.8

= 1.0

25

20

15

Option Price *V*(*S*, 0)

10

5

0

5

30 35 40 45 50 55 60 65 70

Stock Price *S*

**FIGURE 1.** Fuzzy membership function of American call option price at *t* = 0 using FADM.

58

Fuzzy Early Exercise Boundary at Different -levels

Alpha levels

= 0.0

= 0.2

= 0.4

= 0.6

= 0.8

= 1.0

56

Fuzzy Early Exercise Boundary *S* \* (*t*)

54

52

50

48

0.0 0.2 0.4 0.6 0.8 1.0

Time to Maturity *t*

**FIGURE 2.** Fuzzy early exercise boundary *S*˜∗(*t*) at different *α*-levels.

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We consider a triangular fuzzy representation for the parameters involved in the option pricing model. The parameters are chosen as follows

Fuzzy volatility: *σ*˜ = (0*.*15*,* 0*.*20*,* 0*.*25), Fuzzy interest rate: *r*˜ = (0*.*03*,* 0*.*05*,* 0*.*07), Fuzzy initial stock price: *S*˜0 = (48*,* 50*,* 52), Strike price: *K* = 50, Maturity: *T* = 1 year.

The computational domain is discretized using uniform grids, and the *α*-cuts are sampled at increments of 0.1 over the interval [0*,* 1].

The fuzzy solution *V*˜ (*S, t*) obtained through FADM is represented using its *α*-cuts. The resulting membership func-

tions show the range of possible option prices for different levels of confidence (*α*-levels). Figure 1 illustrates the fuzzy price of an American call option at *t* = 0 as a function of the stock price *S*. Figure 2 presents the fuzzy early exercise boundary *S*˜∗(*t*) for various *α*-levels. As *α* decreases, the uncertainty in the boundary widens, indicating that early exercise decisions are also affected by the imprecision in model parameters. The width of the fuzzy price

band increases with the level of uncertainty in the input parameters, demonstrating that fuzzy modeling captures the imprecision inherent in financial markets.

**Comparison with Crisp Models**

To evaluate the performance of FADM under fuzzy uncertainty, we compare the obtained fuzzy price bands with Classical Black–Scholes solution (no early exercise) and Binomial lattice method for American options (crisp).

The results show that the crisp ADM solution lies within the fuzzy price band for *α* = 1, while the binomial method approximates the early exercise premium. The fuzzy model, however, provides a more realistic range of option prices, especially when market conditions are vague or data is incomplete.

This methodology offers an effective tool for risk-averse investors and decision-makers operating in uncertain environments.

# CONCLUSION AND FUTURE WORK

In this study, we proposed a fuzzy analytical framework for the numerical valuation of American options under parameter uncertainty using the Fuzzy Adomian Decomposition Method (FADM). By incorporating fuzzy logic into the classical Black–Scholes model, we were able to capture the imprecise nature of real-world financial parameters such as volatility, interest rate, and asset price using triangular fuzzy numbers. The American option pricing problem, characterized by a free boundary, was handled effectively within the fuzzy setting by solving the model using *α*- cut representations and the recursive structure of ADM. The numerical results demonstrated that the fuzzy pricing

model provides a band of admissible option prices, represented as fuzzy membership functions, rather than a single crisp value. This not only adds robustness to the pricing strategy but also equips investors with a more informed decision-making tool under uncertainty. Furthermore, the fuzzy early exercise boundary revealed that vagueness in input parameters has a significant effect on the optimal exercise strategy.

One can extened this work in to fuzzy dividend-paying assets and American put options and Incorporation of type-2 fuzzy logic or credibility theory for enhanced modeling. This is the future scope of this research work.

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