Basic Concepts of Anisotropy-based Theory for 2D-Systems

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**Abstract.** This paper addresses the analysis problem for two-dimensional linear discrete time-invariant systems describing repetitive processes. An augmented system description is considered for study purposes. For the derived model with an initial state at each cycle, a method for calculating the minimal upper bound of the anisotropic norm is proposed in terms of a system of linear matrix inequalities and a convex constraint. The simulation results are presented, and the key nuances of the calculations are outlined.

Introduction

The study of repetitive processes has become a distinct field in control theory due to the use of two independent time indices (i.e., time shift operators), which introduces alternative analysis methods. Many systems perform a series of cycles or passes, with each pass producing a specific output or profile and having a finite duration [1]. In fields such as electrical engineering, information transmission and processing, robotics, and others, repetitive processes are widespread. Typically, most representations of dynamic processes depend on various variables, such as time, spatial coordinates, iteration number, and information transmission characteristics. After each pass, the system returns to its initial state and begins a new cycle. The objective of such systems is to produce a specific profile during each cycle. Control actions are often employed to reduce fluctuations, but standard control methods cannot address the stabilization problem in this case because they do not account for the two-dimensional nature of the system (i.e., the dynamic properties depend on both the time within each pass and the cycle counter) [2], [3], [4].

Within the framework of linear control, a rigorous theory has been developed to study the stability of systems with two variables interpreted as time, hence the designation as 2D systems. This theory is based on analyzing the properties of a linear operator in the corresponding Banach space [1]. 2D systems include the Roesser model, the Fornasini–Marchesini model, and the repetitive process model [5], [6]. The Roesser model, originating from image processing tasks, describes the dynamics of two components of the state vector — the horizontal and vertical components. The Fornasini–Marchesini model, originally termed a dynamic system with two indices, uses a single state vector and has found applications in the study of two-dimensional digital filters, signal processing, and random failure analysis [7], [8].

With the introduction of iterative learning control, systems with multidimensional dynamics (ND systems) have also been studied. Interest in 2D systems has significantly increased, and iterative control problems remain a focal point [2], [9]. The primary concept of this control strategy is to utilize information from previous passes to enhance accuracy in subsequent cycles.

However, the attenuation of external disturbances in 2D systems within the framework of anisotropy-based theory has not been considered, despite the theory having emerged thirty years ago [10], [11], [12]. Initial ideas for analyzing systems with multidimensional dynamics can be found in [13], some of which are incorporated in this study.

In the case of control, the disturbance is assumed to belong to the class of bounded square-summable signals. This approach is designed to attenuate the worst possible disturbance with limited energy as effectively as possible. However, the controlled plant may be driven by an input signal different from the reference one, making such assumptions often incorrect and the approach not always advisable. Moreover, in theory, robust performance analysis is based on the so-called “worst-case scenario,” resulting in control strategies that can be overly conservative and not always feasible. A concept for utilizing information about the disturbance acting on the system has also been proposed. In control problems, the disturbance belongs to the class of Gaussian noises with a fixed covariance matrix, but the theory suffers from a lack of robustness.

The core concept of anisotropy-based control theory is to measure the difference between an external disturbance and a reference signal. In other words, the distinctive feature of this theory is the selection of methods to describe the external disturbance and the gain from the input to the controlled output of the system, generalizing the formulations of known and control problems. The foundation of this theory comprises the concepts of the anisotropy of a random vector, the mean anisotropy of a sequence of random vectors, and the anisotropic norm of the system. The anisotropy of a random vector measures the deviation of the probability density function (pdf) of a given vector from the Gaussian pdf of a zero-mean white noise sequence. This enables the description of random vectors with unknown stochastic characteristics (yet partially known in the information-theoretical sense) using a scalar-valued function. Over the past three decades, numerous papers have been published on anisotropy-based theory for both time-invariant and time-varying systems [14], [15], [16].

The paper is organized as follows: the second section provides a brief overview of anisotropy-based theory, the third section addresses the problem formulation, the fourth section presents the main result of the paper, the fifth section covers numerical simulation, and the final section contains conclusions.

# PRELIMINARIES

In this section, basic concepts of the anisotropy-based theory are discussed. All the definitions are presented accordingly to [11], [12].

Let us consider a random -dimensional vector with a probability density function (pdf) of the following form:

*,*

where is relative entropy of with respect to the Gaussian random vector with probability density function

where , and is differential entropy of , , denotes expectation. More detailed discuss can be found in [17]. Let us define a vector-fragment of a sequence of random vectors as

The mean anisotropy level for the sequence is defined in [10] as follows:

The anisotropy and mean anisotropy are the information characteristics for the disturbance in time-varying and time-invariant disturbance attenuation problems. Since anisotropy of vector-fragment tends to infinity with the increase of N even for “almost” white noise sequences, for the time-invariant system scenario, it is replaced by mean anisotropy.

In anisotropy-based theory, the cost function is called the anisotropic norm [11], [12]. Let us define a root mean square (RMS) gain as follows:

If transfer matrix of a dynamic system is defined as , then

where notation denotes the space of square summable m-dimensional random vectors. For any positive mean anisotropy level bounded by some , the anisotropic norm of the system is as follows:

The anisotropic norm can unify the -norm and -norm in a special way:

This property holds for non-spherical systems. The anisotropic norm is always between -norm and -norm of the systems and coincides with one of them in limiting cases. Note that the usage of anisotropic norm is reasonable only for the case when system satisfies a strict inequality.

# PROBLEM STATEMENT

Consider an asymptotically stable two-dimensional linear discrete time-invariant system:

where stands the state vector, is the coloured disturbance with a given upper bound a on its mean anisotropy, and is the pass profile. The initial conditions are given and as . Here, *k* denotes the number of the pass, *p* corresponds to time index on each pass time, is the pass length.

The problem discussed here is to obtain the state-space formulas for computation of the anisotropic norm of the system (2) with output coinciding with the pass profile, i.e. .

# MAIN RESULT

First, let us slightly modify the notation for some of the entries in (2) by reducing the lower index by one (while keeping the variable names the same):

To further shorten all the expressions, define two time shift operators and as follows:

Using this notation, the system (2) can be described as follows:

where the extended vectors *,*  and are vectorized forms of the fragments of the vectors , and *,* at any step , respectively:

*,,.*

The initial conditions are now have the form and *.* The matrices in the system (3) are defined as follows:

where denotes the Kronecker product. Based on this, the state and output given by the system (2) can be rewritten in terms of time shift operators as:

where the inverse matrices are understood in the sense of the following time shift operator series (assuming that either are stable matrices or that both of these series are finite):

provided that the action of such an inverse on a vector yields the weighted sum of this vector at all the previous time instants along the time axis corresponding to the shift operator and , respectively. With this in mind, the variable can be eliminated from the system (3), rewriting it as

with the notations

(6)

(7)

Equation (5) provides a convenient form for deriving the main result discussed below.

For the system (5), the so-called worst former filter, widely used in the framework of anisotropy-based theory, is of the form

(8)

where is the Gaussian white noise vector, and the matrices and are those that maximize the root-mean-square gain of the system translating the input to the output , and simultaneously satisfying the equality.

*Theorem 1*: Consider the system (3) and the following matrices

Then, for any number , there exists a unique pair , consisting of the scalar parameter and a positive definite solution of the Riccati equations

*,* (9)

such that

(10)

where is the dimension of the system output, is the controllability Gramian of the former filter (8) satisfying the Lyapunov equation

, (11)

and the matrices and are associated with Riccati equation (9) by the formulas , . Thus, the former filter (8) constructs the worst case input, and the anisotropic norm of the system can be represented as:

(12)

*Proof of the theorem*. The system (3) can be presented as follows:

where the matrices and are constructed according to the operator equations (6) and (7). After, the anisotropic norm of the time-invariant system is derived similarly to [11]. This completes of the proof.

The system of equations (9)-(12) is nonlinear in variables, hence the standard approach of replacing the anisotropic norm calculation problem with a boundedness condition can be applied. Let us formulate the following theorem.

*Theorem 2*: Consider the system (2), describing a repetitive process with arbitrary boundary conditions. Let the additional condition be satisfied, where denotes the spectral radius of . Then, the anisotropic norm of the system is strictly bounded by a given value , i.e., , if the system of inequalities

has a solution with respect to the scalar variables , and the symmetric positive definite positive definite matrices , . This theorem is readily obtained using the result from [14].

Theorem 1 and theorem 2 provide analytic solutions of the anisotropy-based analysis and anisotropic norm boundedness, respectively, for linear discrete time-invariant systems describing repetitive processes.

# numerical experiment

This section reflects the results of numerical simulation of a 2D system with nonzero boundary conditions. A method of numerical solution of the problem is described, tables and graphs of the dependence of the anisotropic norm depending on the parameter are presented.

The problem was solved using the Matlab R2023a application software package using the YALMIP interface, which has additional functions that allow to successfully solve various types of linear matrix inequalities. It should also be noted the use of the standard function geomean, which, in combination with additional packages, allows calculating the determinant as a product of the eigenvalues of the matrix (returns the geometric mean of the eigenvalues of a positive definite matrix). Table 1 shows the values of the a-anisotropic norm as a function of the parameter a depending on the parameter with pass length .

|  |  |
| --- | --- |
| **TABLE 1.** *The values of the anisotropic norm* | |
| **Parameter a** | **Norm value** |
| 0 | 292.3 |
| 1 | 2005.1 |
| 2 | 2616.5 |
| 4 | 3122.3 |
| 8 | 3416.4 |
| 10 | 3455.8 |
| 50 | 3486.3 |

As can be seen from the results, the norm has an upper bound, which is consistent with theoretical expectations. For the system under consideration, the value of the a-anisotropic norm does not exceed 3486.3, i.e. . The graph of the anisotropic norm is shown in Figure 1, which also illustrates its asymptotic behavior.

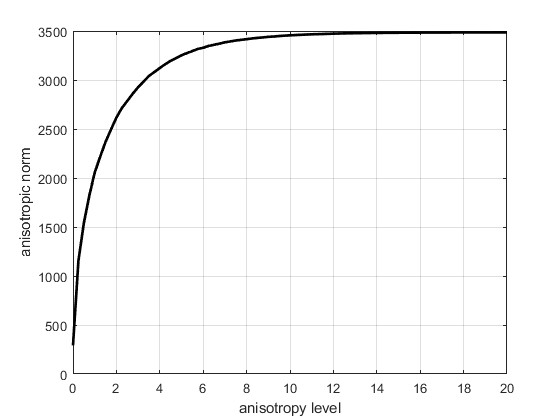


FIGURE 1. Anisotropy norm vs the upper bound of the mean anisotropy *a* with pass length =5.

# CONCLUSION

The forms of recording two-dimensional systems are indicated. Their analysis was carried out: variants of vectorized forms of recording a two-dimensional discrete system with nonzero boundary data were obtained. Auxiliary calculations for calculating the -anisotropic norm using the vectorization process are presented. A computational experiment was conducted, and the results of modeling a two-dimensional stationary system with nonzero boundary data were presented.

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