High-Order Ryabenky Spline-Based Quadrature Formulas for the Numerical Integration of Gastroenterological Signals

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**Abstract.** The article discusses the numerical integration of gastroenterological signs with the help of third-, fifth-, and seventh-order quadrature formulas on the basis of the Ryabenky spline. The study is motivated by the growing need for accurate digital modeling of physiological signals in medical diagnostics. Experimentally derived signal values were approximated using each spline model, and the accuracy of results was assessed through comparative error analysis. Among the models, the seventh-order spline demonstrated superior performance with minimal deviation from the true integral values. These findings confirm the reliability and precision of higher-order spline functions, particularly in capturing the behavior of biomedical signals. The methodology proves effective not only for gastroenterological signal approximation but also suggests potential for broader application in signal processing tasks where high accuracy is essential.

**Keywords:** high-order spline functions, gastroenterological signal, numerical integration, digital signal processing, quadrature formula

# INTRODUCTION

The most challenging thing in the field of medicine nowadays is a pressure on the most effective tools that can help them on the monitoring of their health in real-time and the right and at the right time of diagnosing the diseases that they are trending. Accordingly, numerous cross-cutting researches are currently undertaken in an attempt to address such challenges. Most of the present-day studies define the critical state of medical signal processing and, particularly, the digital processing of biomedical signals as the final solution to the introduction of high-resolution diagnostics and the overall improvement of the quality of the healthcare services which can be offered.

Based on such studies, the analysis of the signals is not only evidence, according to which the flaws in physiological systems can be detected earlier, however, it also brings non-invasive, scalable, and affordable opportunities in the field of diagnosis. This discipline has certainly made itself one of the most smoldering and potent topics of current science. That the subject of interest dwells on human well-being and health is also a great incentive to the scientists, engineers, and clinicians to continue to open new horizons in the different fields, among which are medicine, mathematical modeling, signal processing, biomedical engineering, and artificial intelligence.

Among all the tools under use, the spline functions have proven to be an extremely powerful and smooth solution. As one of the most important tools of approximation theory, the spline functions allow the smooth and confident reconstruction of the signals and have appeared to be actively used in many spheres of science and technologies. The spline-based models are applicable especially in the field of digital signal processing, where the model offers high accuracy and low-cost part of computing and this is very crucial in health care applications. This approach has been identified and affirmed by several scientists in their publications [1, 2, 3, 4, 5] and again shows great functionality.

Besides, the interdisciplinary nature of this field is not in isolation, but is closely woven with a variety of complementary disciplines, such as applied mathematics, numerical analysis, computational techniques and biomedical sciences. Both of them add new input towards the knowledge-building and executable application of methods that Splines entail.

Based on this, the current paper dwells on the use of spline functions to solve a number of fundamental inquiries in medical analysis of signals. In particular, in the following section, a closer look on the construction, the properties and the implementation of constructions such spline functions will be opened.

In cases where the target function is known and well behaved enough and when the purpose is to converge faster, then using splines of high degrees to approximate the target function is not only sensible but beneficial as well. With this purpose in mind, the seventh-degree splines which have been suggested by Ryabenky have been investigated in the details to explore their approximation error and efficiency. By this basis some researchers were able to derive and utilize the 7th order splines of Ryabenky in diverse areas indeed [4, 6, 7, 8], which demonstrates their reliability and versatility even in real life in complex data situations.

In this paper, using quadrature formulas, we will calculate estimates of their error

Let p>0 be a fixed integer, and the trace of a function on a uniform one-dimensional grid [6, 7, 9].

# CONSTRUCTION OF HIGHER-ORDER SPLINE FUNCTIONS

Ryabenkys papers describe the operator L that maps to the function. defined everywhere and being an interpolation spline of degree *2p+1* of defect *p+1*. To estimate the error in the one-dimensional case (1) with respect to the error - and order derivatives, the following estimate follows:

(1)

where the constant depends only on .

It is especially practical that any spline has explicit representation in terms of values, the use of which is especially important in practice. In contrast to Hermitian splines, these splines can also be used to approximate C[a,b] functions. In the works of Sodolev and Ryaben'kiy, for the following expression is given (2) for the function [10, 11].

(2)

where: ; ; general coefficient depending on Stirling numbers (see Eq. 3)

(3)

- k-th order forward difference of ψ at ; - Stirling numbers of the first kind.

This formula constructs the theta coefficients used in Eq. (2)

This is a general form of spline interpolation based on values of a function ψ and computed using Stirling numbers of the first kind. It uses a double summation and symmetrical terms in and , ensuring smoothness.

The relationship between Newtonian and ordinary powers expressions

where and are termed the Stirling numbers of the first and second kinds, respectively

Precisely the following identity (Eq. (4)) expresses the relationship between the power and forward difference operators.

(4)

where:

- finite difference operator; - shift operator, i.e., ;

Also:

(5)

This yields an overall formula of how the finite difference operator can be used together with shift operator. This identity shows how ordinary powers can be reconstructed from falling factorials and finite difference operators (Eq. (5))

;

m=1;

m=2;

general view of Newtonian and ordinary powers

m=k;

m=k+1;

Recursive formula for calculating Stirling -kind: - newton wall; - ordinary wall (connection between them inset connection between numbers Stirling of the 1st and 2nd kind).

, …

So we extract Stirling numbers by matching:

The values of Stirling numbers of the first kind for small m and k are shown in Table 1.

**TABLE 1**. Demonstrative Table of the Stirling Numbers of the First Order Low Orders:

|  |  |  |  |
| --- | --- | --- | --- |
| m | k=1 | k=2 | k=3 |
| 1 | 1 |  |  |
| 2 | -1 | 1 |  |
| 3 | 2 | -3 | 1 |

here the is the Stirling number of the 1st kind;

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Examples for m=1, 2, 3

For m=1:

For m=2:

For m=3:

The expansion of the exp in ordinary powers is using Stirling numbers of the first kind, this gives the coefficients in the previous step, the fraction is an apt device.

The elements . These coefficients played a vital role in the reconstructing of the polynomial representations and paved the way of spline construction. Following on this we now consider the case 𝑝=1, which we explicitly form the spline function using values at nodes and , using finite differences and Stirling weights. This expression is a symmetric and antisymmetric sum of the data point contributions giving to the spline a detailed expression as a sum of the local basis functions (see Eq. (6)), *p=*1

(6)

Using the function values and finite differences:

The spline of the third degree of Ryabenky (7) [2, 4, 5, 9], on the segment

(7)

here: .

Previously, we considered the construction of the third-order Ryabenky spline function [6, 7, 9] and on this basis we will also develop 5th and 7th-order models of the spline function.

In this case, a 5th order spline is constructed (see Eq. (8)) with p=2.

(8)

We compute all combinations of (*k, l*) such that *k+l≤2*:

(9)

In the next step, a 7th-order spline is constructed for the case where p=3 (Eq. (10)).

(10)

The coefficient expressions of the 7th-order spline with different (k, l) combinations are summarized in Table 2.

**TABLE 2.** The coefficient values of a 7th-order spline determined Depending on The Parameters *k,l*:

|  |  |  |
| --- | --- | --- |
| (k, l) | Coefficient | Expression |
| (0, 0) | 6 |  |
| (0, 1) | 6 |  |
| (0, 2) | 3 |  |
| (0, 3) | 1 |  |
| (1, 0) | 24 |  |
| (1, 1) | 24 |  |
| (1, 2) | 12 |  |
| (2, 0) | 60 |  |
| (2, 1) | 60 |  |
| (3, 0) | 120 |  |

(11)

The final form of the 7th-order Ryabenky spline is presented in Eq. (11).

Using the constructed models, the corresponding quadrature formulas were obtained (see formulas (12), (13) and (14)). Now we continue to study the application of quadrature formulas based on the 3rd, 5th and 7th-order Ryabenky spline to gastroenterological signals and present their comparative analysis [12, 13].

(12)

(13)

(14)

here: .

# ANALYSIS AND RESULTS

Table 3 presents the comparative analysis results of the function selected as an experiment on the basis of the spline functions of the 3rd, 5th, and 7th orders given above with integral values at different values of N.

**TABLE 3.** The comparative analysis table of the function based on integral results obtained using 3rd-, 5th-, and 7th-order splines at different values of N

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| N |  |  |  |  | f(x)-S3(x) | f(x)-S5(x) | f(x)-S7(x) |
| 10 | 0,125 | 0,12289827 | 0,12310161 | 0,1239654 | 0,0021000000 | 0,0019000000 | 0,0010300000 |
| 20 | 0,125 | 0,12476026 | 0,124847401 | 0,124955721 | 0,0002400000 | 0,0001530000 | 0,0000443000 |
| 30 | 0,125 | 0,1249311 | 0,124961806 | 0,12499238 | 0,0000689000 | 0,0000382000 | 0,0000076200 |
| 40 | 0,125 | 0,12497137 | 0,124985271 | 0,124997755 | 0,0000286000 | 0,0000147000 | 0,0000022400 |
| 50 | 0,125 | 0,12498548 | 0,124992869 | 0,12499912 | 0,0000145000 | 0,0000071300 | 0,0000008801 |
| 60 | 0,125 | 0,12499165 | 0,124996028 | 0,124999588 | 0,0000083500 | 0,0000039700 | 0,0000004121 |
| 70 | 0,125 | 0,12499476 | 0,124997567 | 0,124999782 | 0,0000052400 | 0,0000024300 | 0,0000002177 |
| 80 | 0,125 | 0,1249965 | 0,124998404 | 0,124999874 | 0,0000035000 | 0,0000016000 | 0,0000001256 |
| 90 | 0,125 | 0,12499755 | 0,124998898 | 0,124999226 | 0,0000024500 | 0,0000011000 | 0,0000007742 |
| 100 | 0,125 | 0,12499822 | 0,124999207 | 0,124999497 | 0,0000017800 | 0,0000007930 | 0,0000005029 |

As demonstrated in the table above, the results obtained through higher-order spline functions are remarkably close to the actual values of the function determined via experimental procedures. This indicates that the selected spline-based model yields accurate and reliable results. The analysis confirms the potential effectiveness of this model, particularly in the medical field-especially in the context of digital signal analysis and disease diagnosis.

In the subsequent phase of the study, the applicability of the proposed model was examined for reconstructing gastroenterological signals, specifically those representing gastric ulcer patterns. The resulting values were comparatively analyzed. In clinical practice, gastroenterological signals play a crucial role in assessing internal organ conditions, such as identifying ulcers and lesions, as well as evaluating the effectiveness and progress of therapeutic interventions.

|  |  |
| --- | --- |
|  |  |

**FIGURE 1**. Segmentation of anomalous boundaries in endoscopic images using spline-based extraction and visualization of gradient values through analytical graphs

These signals serve as an essential source of diagnostic information in the medical domain.

**TABLE 4.** presents the values of gastroenterological signals and their comparative analysis restored on the basis of the higher-order spline function

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| N | f(x) | S3(x) | S5(x) | S7(x) | f(x)-S3(x) | f(x)-S5(x) | f(x)-S7(x) |
| 10 | 0,25 | 0,24975 | 0,2499 | 0,25 | 0,000250000000 | 0,000100000000 | 0,000000000000 |
| 20 | 0,25 | 0,24996875 | 0,2499875 | 0,25 | 0,000031300000 | 0,000012500000 | 0,000000000000 |
| 30 | 0,25 | 0,24999074 | 0,249996296 | 0,25 | 0,000009260000 | 0,000003700000 | 0,000000000000 |
| 40 | 0,25 | 0,24999609 | 0,249998438 | 0,25 | 0,000003910000 | 0,000001560000 | 0,000000000000 |
| 50 | 0,25 | 0,249998 | 0,2499992 | 0,25 | 0,000002000000 | 0,000000800000 | 0,000000000000 |
| 60 | 0,25 | 0,24999884 | 0,249999537 | 0,25 | 0,000001160000 | 0,000000462963 | 0,000000000000 |
| 70 | 0,25 | 0,24999927 | 0,249999708 | 0,25 | 0,000000728863 | 0,000000291545 | 0,000000000000 |
| 80 | 0,25 | 0,24999951 | 0,249999805 | 0,25 | 0,000000488281 | 0,000000195313 | 0,000000000000 |
| 90 | 0,25 | 0,24999966 | 0,249999863 | 0,25 | 0,000000342936 | 0,000000137174 | 0,000000000000 |
| 100 | 0,25 | 0,24999975 | 0,2499999 | 0,25 | 0,000000250000 | 0,000000100000 | 0,000000000000 |

According to the given in Table 4. it can be seen that the quadrature formula made up of the 7th order spline function yielded the best approximations with the values being the nearest to the experimental function. This underscores the effectiveness of higher-order splines particularly the 7th-order spline - in ensuring high precision when digitally processing gastroenterological signals. The numerical values in the table clearly illustrate the model’s performance in accurately reconstructing medical signals. Most notably, the reconstruction errors (|f(x) - S(x)|) were minimal, further validating the practical applicability of this approach.

# CONCLUSION

In this study, spline functions of the 3rd, 5th, and 7th orders were constructed, and corresponding quadrature formulas were developed. A comparative analysis of their integration accuracy was performed. The spline based formulas were employed on approximating the integrals of a gastroenterology signal, which in this case was meant to model a gastric ulcer signal, and compared with experimental values. This can be concluded with the reason being that the maximum approximation error observed in the 3-order spline was 2.5×10-4 but it reduced to 1.0x10-4 in the 5-order spline. On the other hand, the 7th-order spline showed an error that was hardly noticeable thus ranking as the most accurate out of all the tested models.

These results suggest that the spline functions of higher orders can be used to provide great precision in digital signal processing, medical diagnosis, and monitoring. More so, the 7th-order spline model is an effective mathematical solution to real-life problem in applied fields due to its robustness and practicability.

# FUTURE SCOPE

The results of the given research leave a number of perspectives concerning the potential future research and application. Among the most powerful directions, it is possible to note expanding the existing work on multidimensional biomedical information. Although the current study has considered one-dimensional gastroenterological signals, the spline-based quadrature of higher-dimensional datasets (i.e., endoscopic video stream or three-dimensional imaging data at modalities such as MRI, CT) could also be done. It would go far in expanding the usefulness of the technique to clinical diagnosis.

The others possible development is the adaptation of the spline-based integration towards the real time integration. The given algorithms may be implemented in the medical monitoring devices to process physiological signals on-the-fly with additional optimization. This would be especially helpful in a place where real time and accurate perception of signals is vital like a highly care viewing situation or when conducting surgery.

Moreover, the combination of this framework of high-accuracy approximations with machine learning can potentially optimize automatized systems of disease detection. With spline-based integration used as preprocessing step, one can garner cleaner and more accurate representations and representation of the signals which in turn will result in enhanced feature extraction and classification.

It would further be useful to contemplate the strength of these procedures in situations of noise and signal contamination. Because biomedical signals tend to be corrupted by all sorts of interference, it is a natural thing to test the quality of spline-based approximation against these obstacles. This involves trying the hybrid models, which involve the application of splines and filtering methods.

Further, achievement of adaptive spline strategies that seek degree of spline that vary dynamically depending on the behaviour of signal may offer an improved and timely means of integration. These improvements can assist compromise between needs of computational complexity and approximation accuracy.

Lastly, the proposed methodology will have to be extensively validated based on the data of real patients to make sure that the methodology is moved on the path of passing theoretical research into a clinical practice. The collaboration with healthcare institutions would allow conducting clinical trials and making comparative studies that would not only test the accuracy of the models but also introduce them to the diagnostic software tools. This could be done by developing a user-friendly software package, which would implement the spline-based quadrature formulas, such that the method can be used by medical practitioners, researchers and engineers as well.

Overall, the applicability and accuracy of higher-order spline functions as illustrated in this research is indicative of a potential of further growth in a diverse number of biomedical and engineering fields.

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