**Integration of Mathematical and Physical Approaches for Modeling and Optimization of Dynamic System Parameters in Precision Machining of Flexible Shafts**

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**Abstract.** This paper presents an integrated approach to the analysis and optimization of torsional vibrations in a flexible shaft during machining. A mathematical model of a turning process system for a flexible shaft in an elastically deformed state is developed. The system’s dynamics are described using both the Newton–Euler formulation and the second-order Lagrangian method, whose equivalence and respective advantages are demonstrated. Numerical simulation is carried out using the Runge–Kutta method in MATLAB to analyze transient processes. A parametric study examines the influence of stiffness, viscous damping, and inertia on dynamic behavior. To reduce vibration amplitude and transient time, a numerical optimization of key parameters is performed. The results confirm the method’s effectiveness and offer practical recommendations for selecting optimal system parameters. The study highlights the importance of integrated mathematical and physical models, along with numerical methods, in analyzing complex dynamic systems with torsional vibrations.

**Keywords:** mathematical modeling, torsional vibrations, numerical modeling, flexible shaft, Lagrangian mechanics, machining dynamics, optimization

**INTRODUCTION**

Due to growing demands for precision and stability in machining, it is increasingly important to develop mathematical models that accurately describe the dynamics of systems with variable stiffness. The most complex case involves turning flexible shafts, which are prone to deflection, vibration, and elastic deformation. Their behavior must be modeled by differential equations capturing the real-time interaction of the “tool–workpiece–machine” system using measurable and controllable parameters [1].

Torsional vibrations often occur during machining on lathes, drills, and milling machines due to elastic deformation between the driving (chuck) and driven (workpiece, collet) elements. If elasticity and damping are ignored, unwanted oscillations may arise, reducing machining accuracy and equipment lifespan, and possibly causing emergency modes. In modeling low-stiffness shafts in an elastically deformed state, the cutting force primarily induces torsion, while its bending effect is treated as a tool-applied load. Accurate modeling is especially critical when the shaft is aligned along the X-axis and experiences disturbing moments from cutting around the Y and Z axes [2].

Mathematical models of dynamic systems are typically divided into single-mass, two-mass, and multi-mass types [3]. Single-mass models fail to adequately capture the essence of the processes occurring during cutting. Approaches based on models with more than two masses complicate the solution procedure, although solving such models appears to be a promising direction [4]. The analysis of an elastic system accounting for the cutting process, as presented in [5], fundamentally reflects the physical nature of the phenomena occurring within the technological system; however, that study focused on determining natural frequencies without considering external forces.

A review of studies on vibration control shows that many technological system (TS) models inadequately capture underlying processes [6]. They often assume lumped mass and constant stiffness, although in low-stiffness part machining, both parameters vary significantly along the workpiece length depending on tool position.

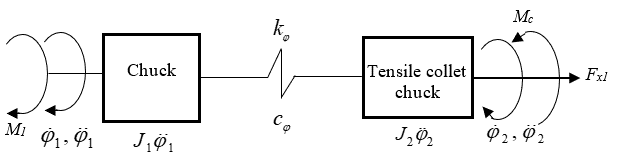
This study develops a dynamic model of two rotating elements connected by elastic and damping links. The system, comprising two masses joined by an elastic shaft, is described using Lagrange equations of the second kind [7]. Numerical simulations via the Runge–Kutta method (MATLAB) analyze transient responses and amplitude-frequency characteristics. Optimization of stiffness, damping, and moments of inertia is performed to minimize vibration amplitude or transient duration. The approach combines rigorous analytical methods with physical interpretation, offering practical recommendations for designing and tuning dynamic systems in mechanical workpiece processing.

**METHODS**

**Mathematical Model of the System**

An elastic-inertial system is considered, consisting of two rotating masses connected by an elastic shaft with damping as shown Figure 1. The first mass represents the chuck (the driving element) with mass , and the second is collet chuck (the driven component) with mass . They are aligned along the axis of shaft rotation and can be approximated as equivalent cylinders. Their moments of inertia *J*1 и *J*2 relative to the rotation axis can be calculated using the formula for a solid cylinder  , where r is the radius of cylinder, *l* is its length. For example, for a chuck with mass =50 кг, radiusbm and length *l*=0.95 m the moment of inertia is approximately  ; for the workpiece with кг, m (a thin rod of the same length ) –. Taken together, the first mass is significantly more inertial than the second. These values will be used in the numerical example.

The shaft possesses a torsional stiffness coefficient  (N·m/rad), which generates a restoring elastic torque  proportional to the relative angular displacement of the components. Additionally, viscous damping with coefficient  (N·m·s/rad) is taken into account. The difference in angular velocities induces a damping torque , which acts against the relative rotation. In the study [5], linear damping is assumed (the resistance force is proportional to the relative velocity), which corresponds to the common viscous friction model. Within the framework of this research, the damping coefficient  is considered dependent on the magnitude of the elastic torque in the shaft [8]. This nonlinear dependence allows to account for increased dissipation under large deformations [9].



**FIGURE 1:** Dynamic model of the technological system

An external torque  can be applied to the first mass — i.e., the chuck—originating from the drive motor. On the second mass, a disturbance torque  may act, representing the cutting resistance from the workpiece. In the general case, the equivalent torque  resulting from cutting resistance can be expressed as the sum:

 (1)

here,  is the steady-state cutting torque, and  is the periodic disturbance torque with frequency  (*М0* is the amplitude of its oscillation relative to the mean value). Such a harmonic component is capable of exciting forced torsional vibrations, particularly if its frequency is close to the natural frequencies of the system-a phenomenon known as resonance.

Let the angular displacements of the lathe chuck and the collet chuck about the shaft axis be denoted by and , respectively (in radians), measured from the equilibrium position. We define the positive direction of rotation as that which causes stretching (torsion, under conditions of longitudinal-transverse bending) of the shaft when is relative to  We apply Newton’s second law for rotational motion to each component. For the first component (the chuck), the sum of torques equals the moment of inertia times the angular acceleration:

 (2)

where – is the external torque from the drive (positive in the direction of ).

It is important to account for the signs: if , then the shaft is twisted and exerts a restoring torque  on the first mass and  on the second (according to the action-reaction principle).

Similarly, for the second mass:

 (3)

In equation (3) appears because  is defined as the resistive torque (opposing the direction of ). Taking into account the driving torque and the resistive torques, the mathematical model can be written as:

 (4)

The system of equations (4) describes the dynamics of a coupled two-mass system. In particular, free vibrations (in the absence of external torques) are represented by the homogeneous system (with the right-hand side equal to zero). The sum of the left-hand sides of equations (2) and (3) yields:

 (5)

In the absence of external torques, this degenerates into the law of conservation of angular momentum:

 (6)

In the absence of damping, such a system has two natural frequencies of oscillation: one that is close to zero (corresponding to a quasi-steady rotation mode with ) and another determined by the torsional stiffness of the shaft (corresponding to an out-of-phase twisting mode). When there is a significant difference between the two masses, one natural frequency becomes very low (effectively corresponding to motion as a single rigid body), while the other becomes high (representing oscillations of the lighter mass relative to the heavier one).

**Lagrange Equations of the Second Kind**

To validate and facilitate the extension of the model, we also employ the Lagrangian approach [10]. We define the kinetic energy, the potential energy  and Rayleigh’s dissipation function оf the considered technological system as follows:

 , ,  (7)

Then, the Lagrange equations with dissipative forces take the form:

(8)

where  is the lagrangian, and  is the generalized non-conservative force (in our case, the external torques). For :

(9)

This reduces to the first equation of system (4). For 

 (10)

which is equivalent to the second equation of system (4). Thus, the Lagrangian method leads to the same equations of motion, confirming their correctness [11, 12].

**Numerical Simulation and Dynamic Analysis**

The resulting system of second-order ordinary differential equations was solved numerically using the classical fourth-order Runge–Kutta method. The MATLAB package (ode45 function) was used with small integration steps to ensure numerical stability at high frequencies. To analyze the system's dynamics, three simulation scenarios were considered:

(I) ‒ free vibrations. Both external torques are set to zero (*M*1 = *M*2 = 0). The masses are initially displaced from equilibrium by an angle , and then released with zero initial angular velocities. The transient process of free torsional oscillations—decaying due to damping—is analyzed.

(II) ‒ forced vibrations at a fixed frequency. A harmonic resistance torque  of a specified frequency , is applied to the second mass, while a constant torque is maintained on the first mass (either by an active drive or set to in a passive scenario). The frequency  is chosen close to the system’s natural frequency to illustrate resonant behavior. The simulation is run until a steady-state oscillation regime is reached.

(III) ‒ amplitude-frequency response (AFR). For a range of frequencies  within a given interval (e.g., 1–80 rad/s), the forced vibration regime is simulated. A harmonic torque of specified amplitude is applied to the system, and after the transient response has decayed, the steady-state oscillation amplitude is recorded. Based on the results, a response curve is constructed showing the dependence of oscillation amplitude on excitation frequency.

The parameters used in the numerical experiments correspond to those mentioned earlier. The shaft stiffness was set to N·m/rad (yielding a natural frequency of 18 rad/s for relative oscillations), unless stated otherwise. The damping coefficient was varied over a wide range to investigate its influence. In particular, for the amplitude–frequency response analysis in scenario (III), relatively low values of  (e.g.,N·m·s/rad) were used to produce a pronounced resonance peak, whereas in the free vibration case, different regimes were analyzed — ranging from weak damping to critical damping.

**Optimization Criteria for System Parameters**

The optimization of dynamic parameters is reduced to selecting appropriate values of  and — if feasible —  in order to achieve a desired compromise between the rate of vibration decay and their amplitude. Two types of objective criteria were considered: minimization of oscillation amplitude and minimization of transient response time.

For steady-state forced oscillations, the objective is to minimize the maximum amplitude of the relative angular displacement  at a given excitation frequency. For transient processes, the goal is to minimize the overshoot ratio (i.e., the ratio of the first peak amplitude to the steady-state value), or ideally, to eliminate oscillatory behavior altogether.

When minimizing the duration of the transient response, the time is evaluated over which the angular displacement difference  approaches and remains within a small neighborhood around zero (e.g., within 1–2% of the initial deviation) without subsequently exceeding that range. This criterion reflects the rapidity of oscillation decay following a disturbance.

Both criteria are closely related to damping. Increasing the damping coefficient  generally reduces oscillation amplitudes and accelerates decay; however, an excessively high value may lead to energy redistribution without oscillations, potentially resulting in a prolonged decay time (the aperiodic transition regime). For given values of  there exists a critical damping level

 (11)

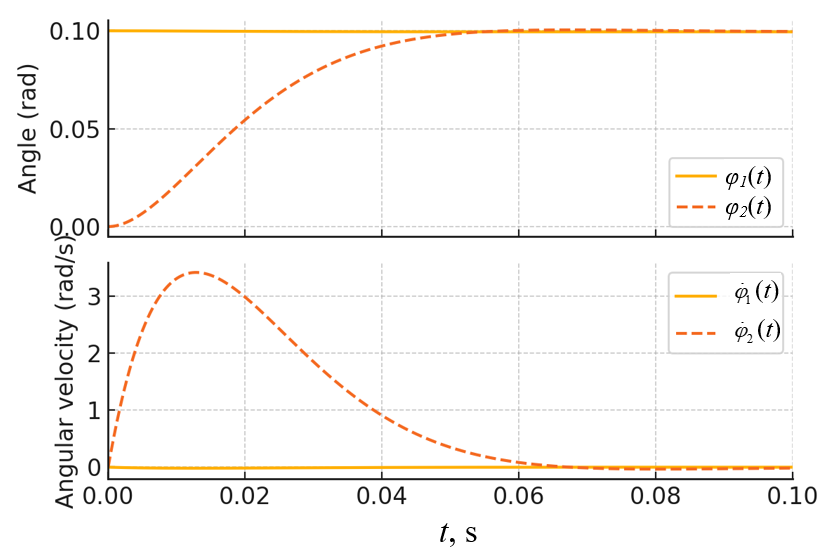
(is the equivalent (reduced) moment of inertia), at which the system transitions from oscillatory to aperiodic behavior. Therefore, the optimum is often achieved near , where oscillations are suppressed (zero overshoot), and the decay time is either minimal or close to its minimum [13]. A parametric analysis was carried out as part of this study:  and  were varied, and and *t*, s were evaluated. The results section below provides illustrations and tables demonstrating the influence of these parameters and the selection of approximately optimal values.

**RESULTS AND DISCUSSION**

**Free Damped Oscillations**

For an initial relative angular displacement of  rad, the system with the parameters described above exhibits damped torsional oscillations. Figure 2 shows a typical transient response for a damping coefficient close to the critical value. It can be seen that the lighter mass quickly “catches up” with the heavier one: the angular displacement (orange dashed curve) approaches (solid line) within a fraction of a second. The damping is strong: the relative angle  drops below 1% of  in approximately 0.06 s. Thus, with sufficient damping, the oscillations are suppressed very quickly — with a characteristic time on the order of half the period of the undamped natural oscillations.

Regimes for different values of *cφ***.** At lower damping, the system enters an oscillatory regime with noticeable overshoot. For example, at  a decaying sinusoidal behavior is observed:  changes sign, and over ~2–3 cycles, the amplitude effectively decays to zero. The overshoot coefficient reaches 50% or more (the first extremum israd for  rad).



**FIGURE 2**. Transient response for free torsional oscillations

As the damping coefficient *cφ* increases, the overshoot amplitude decreases and the duration of oscillations shortens. When , the oscillations vanish entirely (critically damped aperiodic motion), and the settling time reaches its minimum. Further increase in *cφ* (overdamping), while eliminating oscillations, slows down the system’s return to equilibrium. The results are summarized in Table 1.

**TABLE 1.** Effect of damping coefficient on the behavior of the free transient process

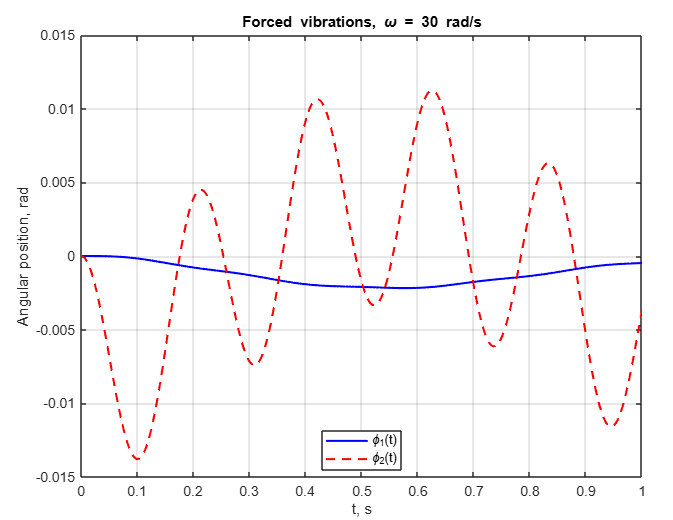
|  |  |  |  |
| --- | --- | --- | --- |
| Damping coef. (N·m·s/rad) | Motion behavior | Overshoot | Estimated settling time *ts* |
| 0 (no damping) | Undamped oscillations | 100% (no decay) | – |
| 10 (≈) | Strongly oscillatory (underdamped) | ~50% | ~0.2  (several cycles) |
| 50 (≈1.0 ) | Critically damped | 0% | ~0.12 (fastest) |
| 80 (≈1.6) | Overdamped | 0% | ~0.22  (slower) |

**Forced Vibrations and Resonance**

To investigate resonance phenomena, the frequency of the external moment was chosen close to the natural frequency of the system. The calculations determined that, in the absence of damping, the natural angular frequency of relative oscillations is

 (12)

(for the given values of и N·m/rad) rad/s, which corresponds to ~2.83 Гц. If the system is subjected to a harmonic resisting torque with a frequency close to *ωn* significant forced oscillations arise (see Figure 3).



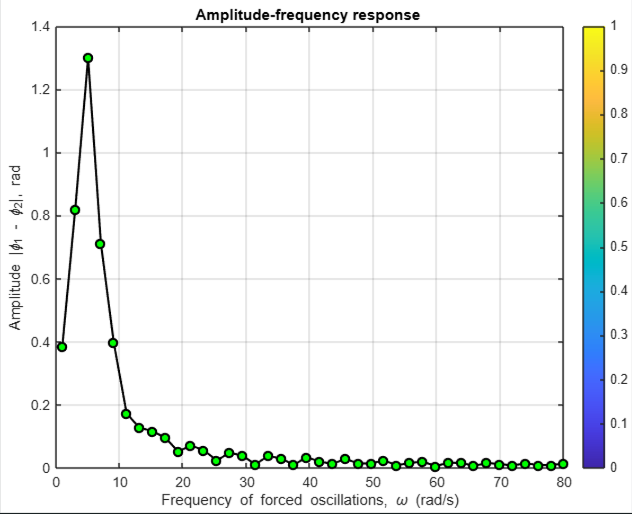
**FIGURE 3**. Scenario (II) – Forced vibrations at a fixed frequency

In the case of low damping , resonance occurs: the steady-state amplitude of the relative angular displacement is large, and the phase of the oscillations lags by approximately 900 behind the external moment (as is typical for resonance in a second-order system). When  is sufficiently large, the resonant amplitude exceeds the neighboring values by only a few percent, meaning the resonance is effectively flattened.

In the steady-state regime, after the initial transient phase, the system settles into a sinusoidal mode in which both masses oscillate at the excitation frequency. The oscillation amplitudes of the chuck and the workpiece differ. Their phases are nearly identical (at relatively low excitation frequencies), except for a slight lag of the workpiece due to damping. At the resonant frequency, the phase shift reaches  relative to the external moment, while at frequencies much higher than the natural one, the oscillations become negligible. These effects are quantitatively evaluated in the amplitude-frequency response (AFR) [14].

**Amplitude-Frequency Characteristic (AFC)**

Figure 4shows the calculated amplitude-frequency characteristic—the dependence of the steady-state amplitude of the relative angular displacement on the frequency of the harmonic torque (with amplitude 1 N·m). A pronounced resonance curve is clearly visible, with a peak around  rad/s, which is consistent with the analytically calculated natural frequency  rad/s (indicated by the red dashed line).



**FIGURE 4.** Scenario (III) – Amplitude-Frequency Characteristic

The resulting AFC provides a practical means to identify hazardous frequencies of external excitation. The frequency response exhibits the typical resonant shape with a passband centered around . The effect of damping manifests as a reduction in the height of the resonance peak and a broadening of the resonant region. Specifically, when  is increased to the critical level, the amplitude at resonance drops by an order of magnitude, and the peak of the AFC shifts several percent toward lower frequencies (which corresponds to an increase in effective damping ) . In the limit of large , the AFC curve becomes flat and lacks a pronounced peak—indicating that the system no longer exhibits strong resonant behavior [15].

**Optimization of System Parameters**

The optimization of the dynamic characteristics of the TS is aimed at minimizing the amplitude of relative angular displacements and reducing the decay time of the transient response. The elastic deformation of the shaft is described by the relative angular displacement

 (13)

which reflects the torsional twist under dynamic loading.

To suppress undesired oscillations and achieve acceptable machining precision, it is necessary to balance the values of the stiffness , the viscous damping coefficient , and the moments of inertia of the componentsand . Increasing reduces the amplitude in the lower frequency range; however, in the absence of sufficient damping, high-frequency disturbances can result in stress amplification within the system. The most effective damping level is typically around , corresponding to the critical damping as equation (11).

Further increase in the damping coefficient  reduces the system’s efficiency due to the lengthening of the transient response and increased energy dissipation [13].

The moments of inertia of the components influence the way energy is distributed within the system. Increasing  makes the system less sensitive to external torques but lowers the natural frequency, which may amplify resonance effects at low excitation frequencies. Achieving a proper balance between  and  allows for an inertia distribution that supports the stability of the oscillatory regime.

The objective function for optimization  is formulated by considering the maximum relative angular displacement ; the decay time of the transient response ; the shear stress ; and the deviation of parameters from technological constraints. The vector of optimized parameters is defined as given in equation (14).

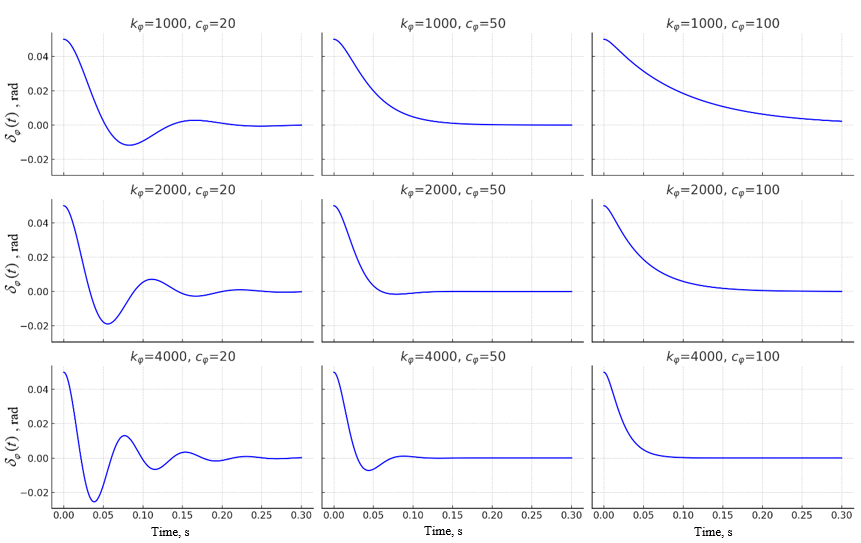
 (14)

Mathematically, the problem is stated as:

 (15)

where  is the relative damping ratio, are weighting coefficients reflecting the priority of accuracy, speed, durability, and mass;  is the allowable elastic deformation (precision requirement). The admissible domain is defined based on strength and dynamic stability constraints.

Parametric simulation results confirm that minimizing vibrations requires a balanced combination of stiffness, damping, and inertia distribution. The graphs shown in Figure 5 illustrate the transient responses of the relative angular displacement for various values of stiffness and viscous damping coefficient . Increasing the stiffness  raises the natural frequency of the system and reduces the amplitude of torsional deflection under low-frequency excitations. However, excessive stiffness can cause an increase in internal forces when subjected to high-frequency inputs. Therefore, the stiffness value must be high enough to prevent elastic deformation, but not excessively large.



**FIGURE 5.** Relative angular displacement  under varying stiffness and 

Increasing the damping  reduces oscillation amplitudes and accelerates their decay. The most significant effect is observed when  is in the range of 0 to ~ 0.2–0.3. At  and =1000 N·m/rad the system achieves an optimal trade-off between response attenuation and amplitude reduction. In this case, the oscillations decay in less than 0.1 s, and the maximum relative angular displacement does not exceed 0.01 rad. In contrast, without damping, the relative rotation exceeds 0.05 rad and does not decay at all (sustained oscillations); at low damping levels, the transient response duration exceeds 0.2 s. Thus, the advantages of parameter optimization are confirmed by numerical analysis. Lower damping values result in either sustained oscillations or prolonged transients, which reduce the system’s dynamic stability.

The inertia distribution also has a significant effect on system dynamics. Increasing *J*1reduces relative oscillations but lowers the natural frequency. When , the system becomes symmetric and dynamically stable.

Given these considerations, optimal vibrational performance is achieved through a coordinated selection of stiffness, damping, and moment of inertia parameters.

The numerical experiments conducted confirm that optimal tuning of system parameters can reduce the amplitude of relative angular displacement by 60–80% and shorten the transient time by more than a factor of two compared to the initial configurations. This supports the effectiveness of integrating mathematical–physical models and numerical optimization methods in tasks involving high-precision machining control.

The obtained results demonstrate the value of an integrated approach to vibration analysis: the combination of a rigorous derivation of the equations of motion and their numerical solution made it possible to identify key patterns and optimal system parameters. The analytical model in the form of equations (4) enabled clear understanding of the influence of key parameters. The stiffness  determines the elastic torque and the system’s natural frequency; the damping coefficient  governs the decay rate; and the ratio of inertias  to  characterizes the energy exchange between the two masses.

The numerical data and plots were obtained using Python/Matlab simulations of the system based on equations (4) and (15) for scenarios I–III.

In studying the technological system (TS) during the machining of flexible shafts, it was found that the uniformity of workpiece motion depends on the magnitude of cutting force torques. Variations in these torques lead to deviations from the target angular velocity and affect transient response time, thereby influencing machining accuracy. For given inertial moments, stiffness, and damping coefficients of the workpiece, the corresponding driving and cutting torques were determined. Optimal values of these parameters were identified, enabling an energy balance between the actuator and the cutting force during machining, which can significantly reduce both material and energy consumption [15].

**CONCLUSIONS**

This study investigates the dynamics of the “lathe chuck–shaft–collet chuck” system with torsional vibrations. A two-mass model with elastic and damping elements was developed, and equations of motion were derived using both Newton’s method and the Lagrange approach, confirming the model’s validity.

Numerical simulations showed that insufficient damping can cause resonance, while optimal damping and high stiffness reduce vibration amplitudes and speed up decay. System parameters critically affect transient behavior.

The influence of moments of inertia and elastodissipative forces on the motion of the workpiece shaft was investigated. Variations in inertia significantly affect angular velocities and accelerations. To reduce their variation range, the shaft’s stiffness and damping coefficients were adjusted. Stiffness strongly influences the amplitude of angular velocity oscillations: increasing it reduces shaft deformation and shortens the transient phase. Higher damping significantly lowers oscillation amplitude while the applied load remains constant. This confirms that both the amplitude and frequency of angular velocity and acceleration oscillations depend on inertia and elastic dissipative properties.

The optimization of parameters made it possible to reduce vibrational effects and improve the system’s stability. The proposed method is applicable for analyzing the dynamics of machine tool assemblies, spindles, and turning processes of flexible shafts in mechanical engineering.

This approach enables the prediction of both transient and steady-state oscillations under various operating conditions, and thus can serve as a foundation for design optimization (e.g., selecting shaft stiffness, damper types, and mass distribution) and for developing vibration suppression control systems [11]. The results are relevant for analyzing torsional vibrations of shafts across a wide range of machinery — from machine tools to transportation and power generation systems [4, 7].

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