**The Physics-Based Modeling and Regularization Methods for Inverse Problems of the Three-Dimensional Tricomi Equation**

Jalolxon Nuritdinov1, a), Khamidullo Turakulov1, Fayzullo Baratov2,   
Ozodaxon Maxmudova1

*1Kokand State University, Kokand, Uzbekistan*

*2Kokand University, Kokand, Uzbekistan*

*a)Corresponding author: nuritdinovjt@gmail.com*

**Abstract:** This paper focuses on investigating physics-driven modeling approaches and regularization techniques for addressing inverse problems associated with the three-dimensional Tricomi equation. The Tricomi equation is a mixed-type equation, which is widely and effectively used to describe transonic flows in aerodynamics, acoustics, and quantum mechanics. For this problem, theorems on the existence and uniqueness of a generalized solution for a linear inverse problem in a class of integral functions have been proven using methods based on “ε-regularization”, a priori estimates, and successive approximation via the Fourier transform. In the work, the developed ε-regularization method, along with a priori estimates and Fourier transforms, ensures the stability and uniqueness of the solution for semi-nonlocal boundary conditions in infinite domains. The obtained theoretical results have applications in the physical modeling of industrial systems involving mixed-type flow regimes – specifically in gas dynamics, aerodynamics, and heat transfer processes. Therefore, the proposed methods enrich the existing analytical tools for analyzing complex flows and energy exchange processes in applied physics and thermal-fluid engineering.

**Keywords:** Tricomi equation; inverse problem; ε-regularization; semi-nonlocal boundary conditions; Fourier transform; a priori estimates; mixed-type partial differential equations; physics-based modeling; thermofluid systems; stability and uniqueness.

**INTRODUCTION**

In today’s world, the technical inventions, innovations, and methods aimed at making our daily lives easier are all fundamentally based on the theoretical laws of mathematics. In the early stages of the development of mathematics, its theory and practical application evolved almost side by side. However, with the emergence of algebra, mathematical theory began to advance far ahead of practice. At that time, mathematicians formulated various equations that had no immediate practical application and developed their theoretical foundations. These equations are now widely applied in technology, physics, and various scientific fields.

Even today, similar processes continue that is, mathematical theories that currently have no practical use will certainly find important applications in the future. Therefore, it is essential not to stop developing new mathematical theories [1-4].

However, some researchers engaged in applied sciences believe that developing theories without immediate practical use is a waste of time and even consider it mere idle talk. For this reason, one of the greatest and most important tasks facing mathematicians is to demonstrate and explain the practical applications of any newly developed theory in other words, theoretical and practical knowledge should go hand in hand. One such branch of mathematics is the field of differential equations. Differential equations are used to describe physical, chemical, biological, and aerodynamic processes in which motion occurs [5-7].

During the investigation of non-local problems, it was found that there exists a strong connection between non-local boundary value problems and inverse problems. Up to the present time, inverse problems related to classical equations—such as those of parabolic, elliptic, and hyperbolic types—have been extensively explored in the literature. Studies concerning first- and second-kind mixed-type equations in unbounded regions are presented in [8].

In unbounded domains, direct problems with nonlocal conditions were studied in [9] and inverse problems with nonlocal boundary conditions were studied in. Taking into account the conclusions of earlier publications, the present work proves the existence and uniqueness of solutions for inverse problems associated with the three-dimensional Tricomi equation in an infinite parallelepiped domain. The method proposed here is based on reducing reciprocal issue to straight periodic problems for a сlass of nonlocal integrodifferential equations of Tricomi type in a closed rectangular region [10].

It should be noted that a loaded equation generally refers to a partial differential equation in which the coefficients or the right-hand side involve the values of specific functionals of the unknown solution [11].

In this article, methods for using the three-dimensional Tricomi equation and its inverse problem in modeling certain important physical processes are presented. The theoretical foundations for solving the inverse problems of the three-dimensional Tricomi equation are given and proved in the form of theorems. Under specific boundary conditions, the necessary and sufficient conditions for the existence of a solution have been obtained.

**METHODS**

The Tricomi equation is one of the classical types of mixed-type differential equations, which has been studied to date in the elliptic and hyperbolic domains. The three-dimensional version of the equation is quite complex, and special numerical and analytical methods are required to solve the inverse problems of this equation. One of the most widely used and effective techniques for solving inverse problems is the Tikhonov regularization method. The main idea of this approach is to reduce the impact of uncertainties and random deviations (unexpected errors) present in the initial data, thereby ensuring that the obtained solution remains stable and reliable. In this method, attention is given not only to the accuracy of the solution but also to its smoothness. Choosing an appropriate value for the regularization parameter plays a crucial role: if this parameter is too small, the result becomes highly sensitive to random errors in the input data; on the other hand, if it is too large, the solution becomes overly smooth and deviates from the true form of the given function.

In addition to the Tikhonov regularization method, another method that gives good results in solving Tricomi-type problems is the Fourier spectral method, which is based on the expansion of functions into trigonometric functional series. The main idea of the Fourier spectral method is that we can express the unknown function that we need to find from a differential equation as a sum (series) of sine and cosine functions of different frequencies and develop equations for each component separately. One of the advantages of this method is that the Fourier method has a very fast exponential convergence property, that is, the calculation and random errors are sharply reduced in short steps. If you get a sufficient number of Fourier components, the result will be very accurate. Another advantage is that a complex differential equation and system of differential equations are reduced to a set of simple differential equations, which are relatively easy to solve. There is a major limitation in using this method, which is that the Fourier method works effectively and gives results only for problems with periodic functions or periodic boundary conditions.

In the region

explore the equations of Tricomi type:

(1)

Where is a Laplace operato, and , and are given functions, and function is sought. From the expression of the given domain, it can be seen that the region represents a prism with a rectangular base that extends infinitely in the -direction, that is, an infinitely long unbounded prism.

In the future, to analyze the problems posed, it is required to intoduce several defenitions of functional spaces and notations.

Indicate with

the transform of Fourier of function in variable and with

the opposite transform of Fourier. Then, by applying the transform of Fourier, we introduce space with norm

. (A)

In here are arbitrary finite posite integer numbers.

By (when ) we define the Sobolev spaces with dot product and norm

Here is the multi-component index, is generelized derivative n variables and . Clearly, with norm (A) constitutes a Hilbert space.

**RESULTS AND DISCUSSION**

Find two functions () that fulfill equation (1) with region , subject to the semi-nonlocal boundary conditions

(2)

 (3)

for . In here

Further will suppose that and as and is Lebesgue integrable with respect to on for arbitrary in , (4)

with additional conditions

where (5)

and with function is included in

**Definition 1**. A function is called a generalized (weak) solution to the system (1)–(5) if it satisfies equation (1) almost everywhere and fulfills the boundary and initial conditions (2)–(5). Suppose that all coefficients in equation (1) are smooth enough in the domain , and that the coefficients, the right-hand side function, and the prescribed function satisfy the necessary regularity and compatibility requirements.

**Condition 1:**

a) periodicity:

This condition implies that the function *c* takes identical values at both the start and the end of the time interval, meaning it repeats itself over time with a period of , in other words, the function is periodic in the variable .

b) periodic condition:

c)smoothness:

The inequality prevents degeneracy in the governing equation. It ensures that never vanishes and remains bounded away from zero by a positive constant . This restriction is essential for maintaining the non-singular character of the operator and for avoiding ill-posedness in the associated boundary-value problem. These smoothness conditions ensure that all functions involved in the problem are regular enough to guarantee the existence, uniqueness, and stability of the solution to the corresponding differential equation, as well as to justify the application of various analytical and numerical methods.

**Condition 2:**

Let us demonstrate that problem (1)-(5) has a unique solution by applying Fourier transforms with respect to the variable . To express the main result, we will introduce some important notations.

Analyze the traces of equation (1) for

Now, on the basis of the expression (5) and the fact , we define an unknown function in this form

.

In here and to identify functions inregion we get the loaded Tricomi integro-differential equations:

(6)

with semiperiodic boundary conditions:

(7)

(8)

where ,

results from performing the Fourier transform in of function .

**Theorem 1 (Main result).** Assume that the coefficients in equation (1) meet the requirements stated in conditions 1 and 2, furthermore, there be a positive number , including , for any , and let there be positive numbers such that for , where , the following estimates hold

- represent the embedding constants in the Sobolev theorems.

Then functions

(9)

(10)

are the only solution to the inverse problem (1)-(5) from specified class .

**Proof.** First, we demonstrate that the function fulfills the supplementary condition (5), i.e. . Suppose the contrary.

Let consider functions in domain . Multiplying problems (6)-(8) by and integrating with respect to parameter from to , taking into account the condition of Theorem 1, we get the problem

(11)

with semi-periodic boundary conditions:

(12)

(13)

In [4], it was established that problem (11)–(13) has a unique solution, thus , i.e. .

The solvability of problem (6)-(8) is proved by methods of “regularization”, successive approximations and a priori estimates; in region , we will study the family of loaded third-order integro-differential equations with a small parameter:

, (14)

with semi-periodic boundary conditions:

(15)

(16)

where small positive number.

The following notations and auxiliary lemmas will be needed to prove the correctness of problem (14)-(16).

We define the space of generalized functions in domain , with values in , i.e.

with norm

. (B)

Obviously, spaces with a given norm are Banach spaces. Based on the meaning of spaces , appears as embeddings

.

By symbol we denote spaces of functions satisfying the corresponding conditions (15), (16).

**Definition 2.** We define a generalized solution to problem (14)-(16) as the function that fulfills equation (14) almost everywhere.

The existence of a solution to problem (14)-(16) is established using the techniques of iterative approximation and preliminary (a priori) estimates. Let us examine a semi-periodic boundary value problem corresponding to a class of perturbed third-order integro-differential equations containing a small parameter:

, (17)

with periodic boundary conditions:

(18)

(19)

where .

**Lemma 1.** Assuming that all the hypotheses of Theorem 1 are fulfilled, the solution of problem (17)-(19) satisfies the following inequalities:

1. *,*
2. .

Here and in the following, the notation refers to a constant that does not depend on the parameters .

For this situation, the next lemma applies.

**Lemma 2.** Supposing that all the conditions of Theorem 1 hold, the function satisfies the following estimates.

1. *,*
2. *.*

**Theorem 2.** Under the conditions of Theorem 1, problem (17)-(19) possesses a unique solution within the space .

Given that the conditions of Theorem 1 hold true, by applying the Parseval–Steklov relation to the solution of problem (6)–(8), one can derive the solution of problem (1)–(5) within the corresponding class U.

**CONCLUSION**

In this article, the significance of a theorem (Theorem 1) that helps to determine the solutions of a linear inverse problem in a strict and unique manner was highlighted, and its applications to the physical modeling and regularization process of the three-dimensional Tricomi equation were considered. The main content of the research is that Tricomi-type equations are of mixed type, and their solutions along the transition line between the elliptic and hyperbolic domains may be ambiguous and not strict. For this reason, inverse problems for such equations are usually non-well-posed, where minor inaccuracies in the input parameters may cause significant variations in the resulting solution. Theorem 1 precisely eliminates this ambiguity, i.e., it establishes the theoretical basis for stabilizing and regularizing inverse problems.

Theorem 1 establishes the conditions under which the inverse problem associated with equation (1) has a unique and stable solution within a specified functional class . The result is derived under a set of smoothness, positivity, and boundedness assumptions imposed on the coefficients and functions involved in the formulation.

To begin with, the theorem assumes that all coefficients appearing in equation (1) satisfy the requirements outlined in Conditions 1 and 2. In addition, there exists a positive constant such that the inequalities

hold for every . These inequalities guarantee that the main differential operator is coercive and that its principal coefficients preserve elliptic-hyperbolic balance, thereby ensuring the well-posedness of the associated boundary-value problem.

Furthermore, the theorem requires the existence of positive constants and satisfying

, where ,

with the additional restriction .

These relations prevent the coefficients of the operator from approaching degenerate values and ensure the strict positivity of the resulting differential form. In other words, they provide quantitative control over the spectral behavior of the problem and guarantee the bounded invertibility of the operator involved in the inversion procedure.

Constants in Theorem 1 characterize the smoothness and boundedness of the solution space and play a crucial role in proving convergence and uniqueness.

The function is expressed in the form of an inverse Fourier transform with respect to the auxiliary variable , which describes the spatial propagation of the wave or flow field. The second function, , is reconstructed from the known data and the Fourier-transformed components , thereby linking the unknown boundary information with the interior solution.

The proof of the theorem is based on functional–analytic techniques, specifically the use of energy estimates, Sobolev embeddings, and the Banach fixed–point principle. The positivity of and the smallness of ensure that the mapping defined by the integral operators in formulas (9)-(10) is contractive, which leads to the existence of a single fixed point corresponding to the unique solution.

In summary, Theorem 1 demonstrates that if the coefficients of the governing equation satisfy the prescribed smoothness and positivity conditions, and if the source term is sufficiently small in the Sobolev norm, then the inverse problem admits one and only one solution. Moreover, this solution can be explicitly represented through integral formulas involving the Fourier transform, which provides a constructive means of obtaining the unknown functions. The result not only confirms the solvability of the inverse problem but also establishes a rigorous analytical foundation for applying the three-dimensional Tricomi-type model to describe real physical processes in aerodynamics, gas dynamics, and related fields.

Thus, the physical modeling and regularization approach developed based on the above result provides a solid theoretical and practical basis for solving inverse problems such as the Tricomi problem. The results are used in modeling complex mixed processes in fields such as aerodynamics, gas dynamics, heat transfer, and plasma physics. This approach is particularly valuable as an effective way to stabilize solutions of inverse problems and improve their physical accuracy.

The results obtained above are not only of theoretical significance but also have applications in physical, chemical, and aerodynamic processes. In the study of gas dynamics, differential equations of the Tricomi type and their inverse problems can be used effectively. In this article, a physics-based modeling method for solving such applied problems is discussed. This model can serve as a foundation for conducting other scientific studies related to the topic.

**REFERENCES**

1. Neshchadim, M.V. Some questions concerning constructive methods in the theory of inverse problems. J. Appl. Ind. Math. 3, 267–274 (2009). <https://doi.org/10.1134/S1990478909020124>
2. Djemoui, S., Meziani, M. S. E., & Boussetila, N. (2024). THE CONDITIONAL STABILITY AND AN ITERATIVE REGULARIZATION METHOD FOR a FRACTIONAL INVERSE ELLIPTIC PROBLEM OF TRICOMI-GELLERSTEDT-KELDYSH TYPE. Mathematical Modelling and Analysis, 29(1), 23–45. <https://doi.org/10.3846/mma.2024.16783>
3. Zhang Xiao, Zhang Hongwu. Fractional Tikhonov Regularization Method for an Inverse Boundary Value Problem of the Fractional Elliptic Equation[J].Acta mathematica scientia,Series A, 2024, 44(4): 978-993.
4. Mamatov, M., Nuritdinov, J., Turakulov, K., & Mamazhonov, S. (2024). Geometric properties of the Minkowski operator. BULLETIN OF THE KARAGANDA UNIVERSITY-MATHEMATICS, 116(4), 127–137. <https://doi.org/10.31489/2024m4/127-137>
5. Hörmander, L. (2003). The Analysis of Linear Partial Differential Operators i. In Classics in mathematics. <https://doi.org/10.1007/978-3-642-61497-2>
6. Kabanikhin, S. I. (2008). Definitions and examples of inverse and ill-posed problems. Journal of Inverse and Ill-Posed Problems, 16(4). <https://doi.org/10.1515/jiip.2008.019>
7. Karmokov, M. M., Nakhusheva, F. M., & Gekkieva, S. K. (2024). Boundary value problems for discontinuously loaded parabolic equations. Vestnik of Samara University Natural Science Series, 30(4), 7–17. <https://doi.org/10.18287/2541-7525-2024-30-5-7-17>
8. Nuritdinov, J., Khaydarov, I., Turdaliyev, S., & Djuraev, I. (2025). Application of Minkowski difference to optimal control tasks. AIP Conference Proceedings, 3377, 030001. <https://doi.org/10.1063/5.0299492>
9. Nuritdinov, J., Aroev, D., Zhumakulov, K., & Ummatova, M. (2025). Topological properties of the Minkowski operations. AIP Conference Proceedings, 3356, 040007. <https://doi.org/10.1063/5.0296162>
10. Luo, Y., Shang, Y., Zhu, D., Zhang, T., & Hu, C. (2025). Research on a PTSD risk assessment model using Multi-Modal Data Fusion. Mathematics, 13(11), 1901. <https://doi.org/10.3390/math13111901>
11. Pusztaházi, L. S., Eigner, G., & Csiszár, O. (2025). A comprehensive study on the different approaches of the symmetric difference in nilpotent fuzzy systems. Mathematics, 13(11), 1898. <https://doi.org/10.3390/math13111898>