**Stress Distribution in a Toroidal Ring Sector   
Subjected to Bending and Twisting**

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**Abstract.** This study is devoted to analytical investigation of stress distribution in a toroidal ring sector with the cross-section of confocal ellipses. Göhner’s method is modified for the present analysis. Pure twisting of the ring sector is first considered. The results for pure twisting and pure bending are combined to reach to the combined loading of the toroidal ring sector. The results obtained are compared with those of packet program ANSYS®. The present method reveals that the predictions of the method are in acceptable range. Stress distribution of curved thin plates is also the outcome of the present method.

**Keywords:** Stress distribution, toroidal, ring sector, Göhner’s method, elliptical cross section, circular cross section, pure torsion, pure bending, combined loading.

**INTRODUCTION**

After it was first noticed in 1910 by Bantlin that curved tubes are more flexible in bending than equivalent straight tubes, the problem of pure bending of curved tubes with thin wall has received great attention by many authors and been extensively investigated by von Karman [1], Thuloup [2], Reissner [3], Clark and Reissner [4] and the others. Recently, Pala [5] has investigated the similar pure bending problem using Göhner’s method [6] based on an extensively different approach from those developed for curved solid tubes of circular cross-section, with an exception that the cross-section of the tube has been considered elliptical [7].

Although there are vast number of papers on pure bending or pure bending and internal pressure, only few papers are available on the problem of pure twisting and combined loading of tubes, that is bending and twisting of curved tubes. This paper is devoted to analyzing the internal stress distribution of a toroidal ring sector with the cross-section of two confocal ellipses subjected to simultaneous bending and twisting. First, the pure torsion of the ring sector is considered using the method of successive approximation with some modification utilized by Göhner for helixal springs. Second, the results given by Pala for pure bending in the case of elliptical cross-section are modified for the present problem. Third, the results are combined in a suitable manner since the related equations are all second order and linear.

The reason for selecting an elliptical cross-section is twofold. In the case of circular cross-section composed of two confocal circles, the cross-section of the neck would tend to deforming nearly into an ellipse for the case of small displacement. After an experimental measurement, the lengths of the principal axes of ellipses can be evaluated in the analytical results of the method. In addition, toroidal rings of elliptical cross-section can also be analyzed in the present case.

Solid circular sections and thin rectangular plates subjected to bending, twisting and combined loading are two special cases of the method derived from results for elliptical solid cross-section [8-10].

**ANALYSIS**

**A. Pure twisting of ring sector**

Let us consider a toroidal ring sector subjected to a couple force *P* perpendicular to the ring plane and passing through the center of the ring (see Fig. 1a). These forces produce a moment  at every section of the toroidal ring. Related dimensions are shown in Fig. 1b. Here, the lengths of the principal axes of internal elliptical hole are related to those of outer ellipse such that *c*=*ka, d=kb* due to the reason that will be explained later.

|  |  |
| --- | --- |
|  |  |
| a) | b) |

**FIGURE 1.** Toroidal ring sector subjected to pure twisting.

Cylindrical coordinates will be utilized for the solution. In the case of pure torsion, assuming that only shear stress  and  are different than zero, the equation of equilibrium gives

 (1)

The next two equations are identically satisfied [14], we write

 (2)

Substituting Eqs. (2) into the compatibility equations [7] and realizing that  we have

 (3a)

 (3b)

Taking into account the expansion

 (4)

shear stresses for the first approximation can be written as

 (5)

In the same manner, we obtain for the second and third approximation that [7]

 (6)

and

 (7)

**B. Pure twisting of ring sector with the cross-section of confocal ellipses**

We can consider a stress function in the form of

 (8)

composed of the equation of boundary surfaces because the function  is zero on the inner and outer surfaces. Here, the coefficients *a*, *b* and *c*, *d* are the lengths of semi axes of the ellipses. However, the form does not give way to using such a form since we encounter with the situation that the *k* initially assumed to be constant would be dependent upon the variables  and . Therefore, we have to choose a stress function which is of the form

 (9)

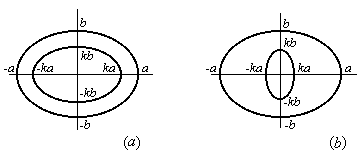
that is zero only on the outer surface. Substitution of this expression gives

 (10)

Thus, the function

 (11)

satisfies the necessary condition on the surface.



**FIGURE 2.** Cross-section of confocal ellipses.

Now, consider an ellipse whose equation is given by

 (12)

where *k* is a constant. This curve is confocal with the outer boundary curve. As can be seen from Eq. (11),  is zero also on this curve, and therefore this ellipse is a stress line for . Shear stress of any point on this curve is tangent to the curve. Now, if a toroidal surface is passed from this stress line, no stress acts upon this surface. Thus, we may consider that the material of this space enclosed by this surface is removed. As a result, the stress function given by Eq. (11) can also be applied to a toroidal section including an elliptical hole.

In the meantime, an attention should be paid to the restriction arrived. The present method is no valid for a cross-section plotted in Fig. 2. *b* since inner boundary is not a stress line in this case.

Now, using Eq. (11), we obtain from Eqs. (5) the first approximation for stress components as

 (13 *a*)

 (13 *b*)

The twisting moment corresponding to this distribution is of the form

 (14)

from which  is obtained as

 (15)

For a given ratio of rotation (), the stresses in a hollow ring sector become the same as those in a solid ring sector, with the exception that the moment *Mt* decrease at a rate of 1-*k*4: . Thus, the equation

 (16)

is also valid for an elliptical hole. Here,  is given by Eq. (15).

Now, in order to obtain a second approximation, we have

 (17)

The solution which satisfy Eq. (17) and the boundary condition  may be assumed in the form

 (18)

Substitution of Eq. (17) in Eq. (16) gives

 (19)

Thus, Eq. (18) takes the form

 (20)

If we substitute this stress function into Eqs. (32), we have the second approximation for stresses as

 (21 *a*)

 (21 *b*)

The moment for this case is then obtained as

 (22)

Where



To obtain the third approximation, we rewrite

 (23)

The stress function  that satisfies Eq. (36) is assumed to have the form

 (24)

where *a*1, *a*2 and *a*3 are constants to be determined. By utilizing the form of  in Eq. (23) and equating the equal powers of  and , it is not difficult to show that

 (24 *a*)

 (24 *b*)

 (24 *c*)

Now, using the form of  with known constants *a*1, *a*2 and *a*3 in Eqs. (7) gives

 (25 *a*)

 (25 *b*)

**C. Pure bending of toroidal ring sector with the cross-section of confocal ellipses**

The problem of pure bending of a toroidal ring sector with cross-section of confocal ellipses were studied by Pala [5]. Assuming for a ring sector subjected to couple as shown in Fig. 3 that only non-zero stresses are , ,  and , we have for a symmetrical stress distribution that

 (26)

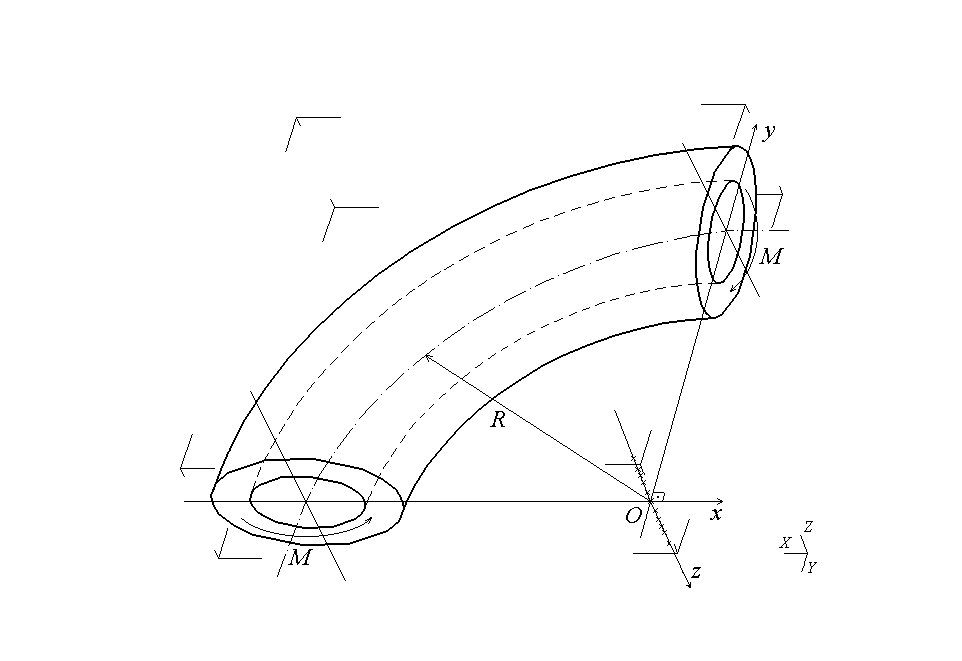
 (27)

 (28)

 (29)

where

, , , , , ,, , ,



**FIGURE 3.** Pure bending of toroidal ring sector with elliptical cross-section.

 [5]. Stress function ∅ [5] were assumed to have the form

 (30)

**D. Toroidal ring sector subjected to combined loading**

Assume that the toroidal ring sector is loaded as in Fig. 4a. For small displacements, the results of the present method and Eqs. (26)-(29) can be superposed in a suitable manner, because equations solved are all of second order in both cases. However, when the constants assumed in the theoretical analysis are considered, this combination of stresses can be made only for cross-section shown in Fig. 4. *b* and *c*. We note that the lengths of principal axes of the inner elliptical hole are chosen such that , , where *k* is a constant.

Thus, combining the results of pure twisting for the first approximation and Eqs. (26)-(29) for pure bending, the stresses are obtained as follows:

;

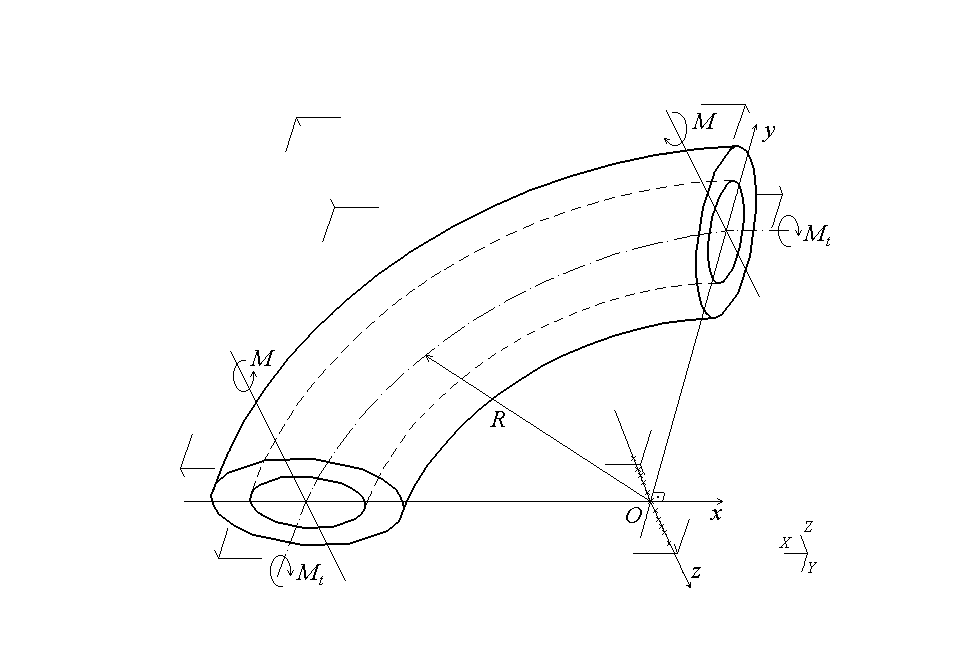
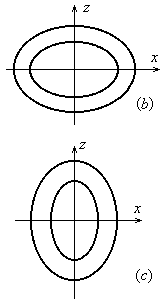
;

;

 (31)

;

;

(*a*)

**FIGURE 4.** Toroidal ring sector subjected to combined loading.

**RESULTS**

The third and perhaps the most important reason for a elliptical cross-section is to predict some analytical results for thin plates and circular pipes subjected to pure twisting, bending and combined loading, that is twisting and bending.

The results for the solid elliptical cross-section (*ka=kb*=0, where *k*=0) are as follows:

;

;

;

 (32)

;

;

, , , , ,

.

Here, in deriving the first two equations of Eqs. (32), we made use of the stress function given by Eq. (11) and substituted this form into Eq. (5) for the first approximation. From third to fifth in Eqs. (32), the form of the stress function were taken as

 (33)

in the case of solid elliptical cross-section. Only this form satisfies the equation of equilibrium and the compatibility equations. The constants *c*1 and *c*2 in the expression for  are found from compatibility equations. For the derivation of the value of *A*0 [5].

In the same manner, the results for the cross-section of confocal circles (*a=b, ka=kb*) can be found by simply taking





 (34)







, , , , , .

where *c*1 and *c*2 are obtained from compatibility equations (see., Ref. [5]). It should be noted that the stress function for twisting and bending are given by

 (35)

and

 (36)

respectively.

The results for solid cross-section of circular boundary (*a=b, ka=kb=*0) are obtained as follows:







 (37)





, , , , , .

In this case, the stress function for twisting is

 (38)

while the stress function for bending is

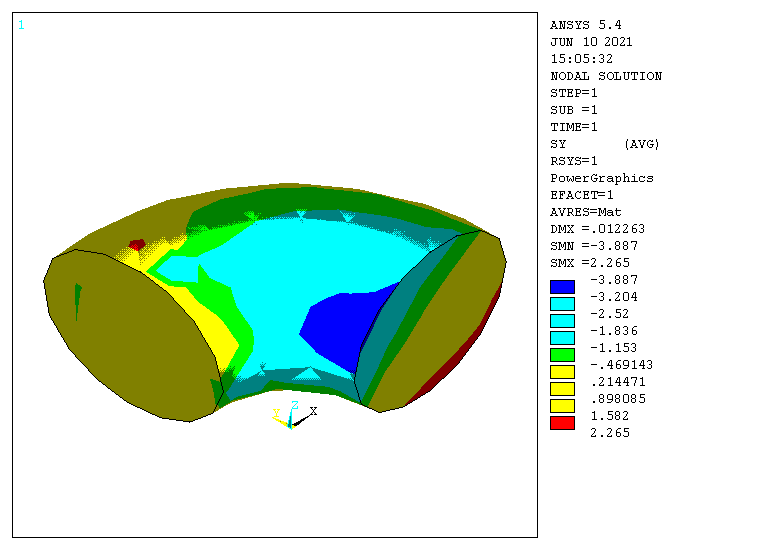
 (39)

In order to show whether the predictions of the present method is in acceptable range, the same toroidal ring sector has been analyzed using ANSYS®. Model drawn on ANSYS® packet program is shown in Fig. 5. External forces for bending are applied at four points as shown in Fig. 5. It is assumed that one end does not make any displacement.

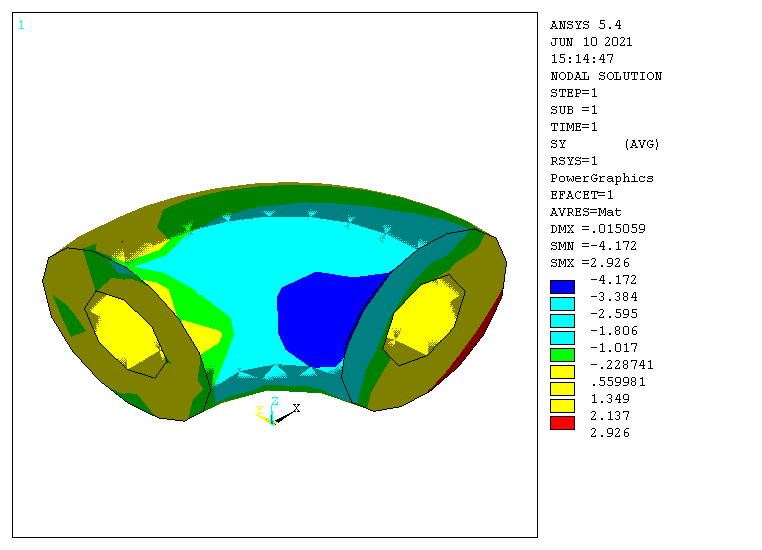


**FIGURE 5.** Computer modeling of elliptical ring sector.

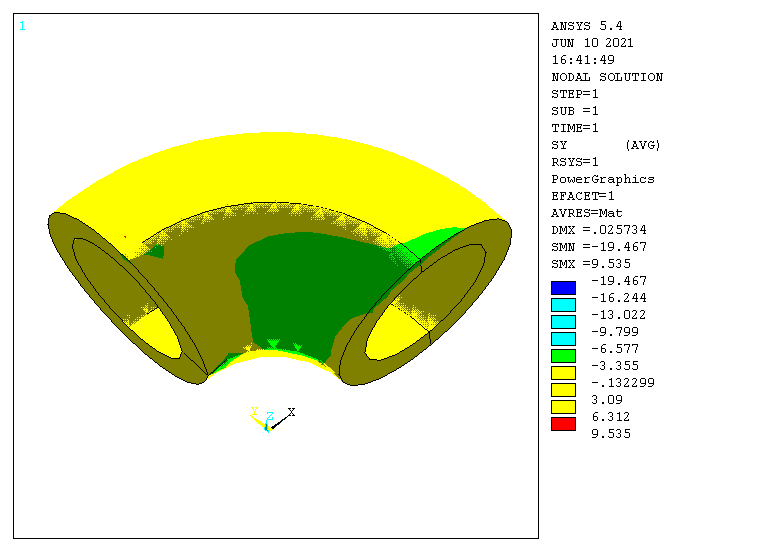
The results for  in the case of pure bending are shown in Figs. 6, 7 and 8 for various values of *k*’s.



**FIGURE 6.** ANSYS® results for  in the case of pure bending (*k*=0).



**FIGURE 7**. ANSYS® results for  in the case of pure bending (*k*=0.5).



**FIGURE 8.** ANSYS® results for  in the case of pure bending (*k*=0.7).

**DISCUSSION**

The related dimensions are given in Table 1. Table 2 shows the comparison of ANSYS® results with analytical results at the points ,  and , .

**TABLE 1.** Dimensions and material properties of the model.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| **Model (Isotropic)** | | ***k=*0**  **[*mm*]** | ***k=*0.5**  **[*mm*]** | ***k=*0.7**  **[*mm*]** | **Poisson Ratio**  **(ν)** | **Modulus of elasticity**  **(*E*), [*Nmm*-2]** | ***F*[*N*]** |
| **Ring Sector of** | ***a*** | 137.5 | 137.5 | 137.5 |  |  |  |
| **Circular** | ***b*** | 137.5 | 137.5 | 137.5 |  |  |  |
| **Cross-Section** | ***c*** | 0 | 68.75 | 96.25 |  |  |  |
| **(*R*=250*mm*)** | ***d*** | 0 | 68.75 | 96.25 |  |  |  |
| **Ring Sector of** | ***a*** | 137.5 | 137.5 | 137.5 | 0.28 | 2.0e5 | 9810 |
| **Elliptical** | ***b*** | 110 | 110 | 110 |  |  |  |
| **Cross-Section** | ***c*** | 0 | 68.78 | 96.25 |  |  |  |
| **(*R*=250*mm*)** | ***d*** | 0 | 55 | 77 |  |  |  |

**TABLE 2.** Comparison of theoretical and numerical results.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Toroidal Ring** | **Constants** | | ***k*=0** | ***k*=0.5** | ***k*=0.7** |
| (*R*=250*mm*) |
|  | *c*0 | | 20411.885 | 13054.978 | 6737.457 |
| **E** | *A*0, [*mm-*4] | | -7.239e-9 | -2.89e-8 | -1.48e-8 |
| **l** | *c*1 | | -1.084 | -1.0865 | -1.0824 |
| **l** | *c*2 | | 0.195 | 0.1923 | 0.196 |
| **i** | *cE*, [*N/mm*5] | | 0.0109 | 0.0117 | 0.0144 |
| **p** | **Theoretical Results** | | | | |
| **t**  **i** |  | =137.5*mm,*  =0 | -1.502 | -1.959 | -2.771 |
| **c**  **a** | [*N/mm*2] | =-137.5*mm*,  =0 | 1.495 | 1.258 | 1.189 |
| **l** | **Computational Results (ANSYS®)** | | | | |
| **cross-section** |  | =137.5*mm*,  =0 | -1.836÷  -1.153 | -1.806÷  -1.017 | -3.355÷  -0.132299 |
|  | [*N/mm*2] | =-137.5*mm*,  =0 | 0.898085÷1.582 | 0.559981÷1.349 | 0.132299÷3.09 |

It is easily seen from Table 2 that the predictions of the present method are in good agreement with numerical ones. For example, at the point =137.5 *mm*, =0 theoretical values of  gives =-1.502 *Nmm*-2 while ANSYS® program gives a value between -1.836 *Nmm*-2 and 1.153 *Nmm*-2 for . What is observed in Table 2 is that average values of computational results are generally less than theoretical results.

**CONCLUSION**

In the study, the stress distribution in a toroidal ring sector with a cross-section of confocal ellipses was calculated analytically using the Göhner method.

The results of the pure torsion and pure bending in the toroidal ring sector were combined. The obtained analytical results were compared with the stresses in the toroidal ring sector created in the ANSYS program.

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