Vertex odd power mean labeling of families of graphs

A Sasikala1,a), V. Poovila1,b), C. Vimala1,c)

1Periyar Maniammai Institute of Science & Technology (Deemed to be University), Vallam, Thanjavur, Tamil Nadu, India.

Corresponding author: a)[sasikala@pmu.edu](mailto:sasikala@pmu.edu), b)[pudurpoovila@gmail.com](mailto:pudurpoovila@gmail.com), c)[vimalasakthi@pmu.edu](mailto:vimalasakthi@pmu.edu)

**Abstract:** Graph labeling plays a significant role in graph theory, where labels are assigned to the vertices, edges, or both, under specific conditions. If the graph G (V, E) with p vertices and q edges is feasible to label the vertices x ε V with different labelings f(x) from f :V(G) →{1, 3, …, 2q + 1} in such a way that when each edge e = uv is labeled with

f\*(e = uv) = or f\*(e = uv) = and the edge labelings are distinct. This labeling is called vertex odd power mean labeling*.* In this paper it is proved that the disconnected cycles graph sCr, the Subdivision graphs Ps,r and S(Ks,r), Firecracker graph, grid graph, Spanning Tree, the graphs Pr ʘ Ks, and Pr + sSr satisfy the vertex odd power mean labeling.

**Keywords:** Disconnected cycles graph, Subdivision graphs, vertex odd power mean labeling

# Introduction

Graph labeling is an established area in graph theory, involving the assignment of integers to the vertices, edges, or both, under specific conditions [1]. The concept of graph labeling, first introduced in the mid-’60s, has gained significant attention due to its wide range applications. The past six decades, more than 350 various types of labeling techniques have been introduced by several authors, including graceful labeling, odd graceful and even graceful labeling, harmonious labeling, felicitous labeling and mean labeling etc. and almost 3600 articles have been published [2].

Graph labeling has been motivated by its applications in various fields like network design, coding theory, and communication systems etc, and its intrinsic mathematical interest and relevance to practical problems.

In this paper we focus on the vertex odd power mean labeling and apply this for certain families of graphs.

# Definitions

## Definition 2.1: Graceful Labeling

A *graceful labeling* of a finite undirected graph G = (V, E), is a one – to – one mapping f from the set V of vertices of G to the set {0, 1, 2, ..., m}, where m = |E|, such that the induced edge labels are all distinct. Given f, the “induced edge labeling” is one – to - one mapping f\* from the set of edges of G to the set {1, 2, ..., m} given by f\*(uv) = |f(u) − f(v)| [3].

## Definition 2.2: Harmonious Labeling

A *harmonious labeling* on a graph G is an injection from. the vertices of G to the group of integers modulo k, where k is the number of edges of G, that induces a bijection between the edges of G and the numbers modulo k by taking the edge [4].

## Definition 2.3: Felicitous Labeling

A simple graph G is said to be *felicitous* if there exists a one–to–one function f: V(G) → {0,1, 2, ..., q} such that the set of induced edge labels f\*(uv) = (f(u) + f(v)) (mod q) are all distinct [5].

## Definition 2.4: Mean Labeling

A graph G with p vertices and q edges is called a *mean graph* if there is an injective function f from the vertices of G to {0,1,2,...,q} such that when each edge *uv* is labeled with if f(u) + f(v) is even, and if f(u) + f(v) is odd, then the resulting edge labels are distinct [6].

## Definition 2.5: Power Mean Labeling

The graph G (V, E) with p vertices and q edges, if it is feasible to label the vertices x ε V with different labelings f(x) from f :V(G) →{1, 2, …, q + 1}in such a way that when each edge e = uv is labeled with

f\*(e = uv) = 

or

f\*(e = uv) = 

and the edge labelings are distinct. This labeling is called *power mean labeling* [7].

## Definition 2.6: Power Exponential Mean Labeling

A (p, q) graph G is said to be a *power exponential mean graph* if there exist a one to one correspondence f: V ⟶ {1,2,…,p} such that induced function f\*:E(G)⟶N given by

f\*(e = uv) = 

or

f\*(e = uv) = 

for every uv ϵ E(G) are all distinct [8].

## Definition 2.7: Power 3 Mean Labeling

A function f is called a power 3 mean labeling of a graph G = (V, E) with p vertices and q edges if it is possible to label the vertices 𝑥 ∈ 𝑉 with distinct labels f(x) from 1, 2, …., q + 1 in such a way that when each edge e = *uv* is labeled with

f\*(e = uv) = 

or

f\*(e = uv) = .

Then the edge labels are distinct [9].

## Definition 2.8: Disconnected Cycles mC3, mC4,…, mCn

The *graph mCn* is a disconnected graph G consisting of n copies of the graph Cn [10].

## Definition 2.9: Subdivision Graph

For a graph G, the Subdivision graph of G, denoted by S(G), is the graph obtained from G by inserting a new vertex in each edge of G [10].

## Definition 2.10: Firecracker Graph

A *Firecracker FC(1n,K1,m)* as a graph obtained from the concatenation of star 1, by linking one leaf from each. Extended fire cracker is the graph obtained from the concatenation of stars by linking a vertex to one leaf from each of any number of stars 1, by a path [10].

## Definition 2.11: Grid Graph

A two-dimensional grid graph, also known as a rectangular grid graph or two-dimensional lattice graph, is an m × n lattice graph that is the graph Cartesian product  Pm □ Pn of path graphs on m and n vertices. The *m × n grid graph* is sometimes denoted by L(m, n) [10] .

## Definition 2.12: Spanning Tree

A *spanning tree* is a connected graph using all vertices of the graph G in which there are no circuits. In other words, there is a path from any vertex to any other vertex, but no circuits [10].

## Definition 2.13: Graph Pn ʘ Km

The *graph Pn ʘ Km* is a connected graph obtained by joining a ‘n’ number of single pendant edge to each vertex of a path.

## Definition 2.14: Graph Pn + mSn

The *graph Pn + mSn* is a connected graph obtained by joining a n number of star K1,n to each vertex of a path.

# Results

## Vertex Odd Power Mean Labeling

The graph G(V, E) with p vertices and q edges, if it is sufficient to label the vertices x ε V with different labelings f(x) from odd numbers of 1, 3,…., 2q + 1 in such a way that when each edge e = uv is labeled with

f\*(e = uv) = 

or

f\*(e = uv) = 

and the edge labelings are distinct. This labeling is called ***vertex odd power mean labeling*** [11]. The vertex odd power mean labeling is defined by C. Vimala and B. Kavitha in 2020 [11]. They proved that cycle, star, connected graph Cn + 2P2 are vertex odd power mean graphs. This study examines the concept of vertex odd power mean labeling and analyzes its applicability to the graphs such as the disconnected cycles graph sCr, the Subdivision graphs Ps,r and S(Ks,r), Firecracker graph, grid graph, Spanning Tree, the connected graphs Pr ʘ Ks, and Pr + sSr .

**Theorem 3.1:** A disconnected cycle graphs sCr admits vertex odd power mean labeling.

**Proof:** Thedisconnected cycle sCr has *sr* vertices and *sr* edges.

Define the vertex labeling f: V(sCr) →{1, 3, …, 2q +1} as

**v1ʹ v2ʹ vrʹ**

**v1 v2 vr**

**v1s-1 v1s-2 v2s-1 v2s-2 vrs-1 vrs-2**

Fig. 1: Disconnected of cycle sCr

f(vi) = 2ri – 2r + 1 for 1 ≤ i ≤ r

f(vi(1)) = 2ri – 2r + 3 for 1 ≤ i ≤ r

f(vi(2)) = 2ri – 2r + 5 for 1 ≤ i ≤ r

…….

f(vi(s-2)) = 2ri - 3 for 1 ≤ i ≤ r

f(vi(s-1)) = 2ri - 1 for 1 ≤ i ≤ r

Consequently, the vertex labels are distinct and the disconnected cycle sCr satisfies the vertex odd power mean labeling condition. As a result, the graph is a vertex odd power mean graph.

**Theorem 3.2:** *The Subdivision graph S(Ks,r) admits vertex odd power mean labeling*.

**Proof:** The Subdivision graph S(Ks,r)has *sr + 1* vertices and *sr* edges.

Define the vertex labeling f: V(S(Ks,r)) →{1, 3, …, 2q +1} as

For 1 ≤ i ≤ r

f(vi) = 2i – 1

f(vi(1)) = f(vr) + 2i

f(vi(2)) = f(vr(1)) + 2i

…….

f(vi(s-2)) = f(vr(s-3)) + 2i

f(vi(s-1)) = f(vr(s-2)) + 2i

f(w1) = f(vr(s-1)) + 2.

Thus the vertex labels are distinct and the Subdivision graph S(Ks,r)satisfies the vertex odd power mean labeling condition. Therefore, the graph is a vertex odd power mean graph.

**vr vr-1 v2 v1**

**w1 vrʹvr-1ʹv2ʹv1ʹ**

**vrs-1vr-1s-1v2s-1v1s-1**

Fig. 2: Subdivision graph S(Ks,r)

**Theorem 3.3:** The Firecracker graph FC(1r, K1,s) admits vertex odd power mean labeling.

**Proof:** The Firecracker graph FC(1r,K1,s) has rs + 1 vertices and sr edges.

Define the vertex labeling f: V(FC(1r,K1,s)) →{1, 3, …, 2q +1 } as

For 1 ≤ i ≤ (s + 1)r – 1

f(vi) = 2i – 1; i ≠ s + 1, 2s + 2, 3s + 3, …, (s + 1) (r – 1)

For 1 ≤ i ≤ r

f(viʹ) = 2si – 1

f(u1) = f(vrʹ) + 2

Thus the vertex labels are distinct and the Firecracker graph FC(1r,K1,s) satisfies the vertex odd power mean labeling condition. Hence, the graph is a vertex odd power mean graph.

**Theorem 3.4:** The Subdivision Graph Ps,r admits vertex odd power mean labeling.

**Proof:** The Subdivision Graph of Ps,r has rs + 2 vertices and rs + s edges.

Define the vertex labeling f: V(Ps,r) →{1, 3, …, 2q +1} as

For 1 ≤ i ≤ r + 2

f(vi) = 2i – 1

f(vi(1)) = f(vr) + 2i

f(vi(2)) = f(vr(1)) + 2i

…….

f(vi(s-2)) = f(vr(s-3)) + 2i

f(vi(s-1)) = f(vr(s-2)) + 2i

f(uj) = f(vr(s-1)) + 2j for j = 1, 2

**v1 v2 vr-1 vr**

**u1 u2**

**v1s-1v2s-1vr-1s-1vrs-1**

Fig. 3: Subdivision Graph Ps,r

Thus the vertex labels are distinct and the Subdivision Graph of Ps,r satisfies the vertex odd power mean labeling conditions. Therefore, the graph is a vertex odd power mean graph.

**Theorem 3.5:** The grid graph admits vertex odd power mean labeling.

**Proof:** The grid graph has 2sr - 1 vertices and 2s (r – 1) + 3 edges.

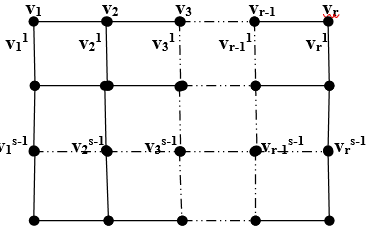
****

Fig. 4: Grid graph

Define the vertex labeling f: V(Gr) →{1, 3, …, 2q +1} as

For 1 ≤ i ≤ r

f(vi) = 2s(i – 1) + 1

f(vi(1)) = 2s(i – 1) + 3

f(vi(2)) = 2s(i – 1) + 5

…….

f(vi(s-2)) = 2si – 3

f(vi(s-1)) = 2si – 1

Thus the vertex labels are distinct and the Grid graph satisfies the vertex odd power mean labeling condition.Therefore, the graph is a vertex odd power mean graph.

**Theorem 3.6:** *The Spanning Tree admits vertex odd power mean labeling*.

**Proof:** The Spanning Treehas *sr* vertices and *sr* - 1 edges.

Define the vertex labeling f: V(STs,r) →{1, 3, …, 2q +1} as

For 1 ≤ i ≤ r

f(vi) = 2i – 1

f(vi(1)) = f(vr) + 2i

f(vi(2)) = f(vr(1)) + 2i

…….

f(vi(s-2)) = f(vr(s-3)) + 2i

f(vi(s-1)) = f(vr(s-2)) + 2i

Thus the vertex labels are distinct and the spanning treesatisfies the vertex odd power mean labeling condition. Therefore, the graph is a vertex odd power mean graph.

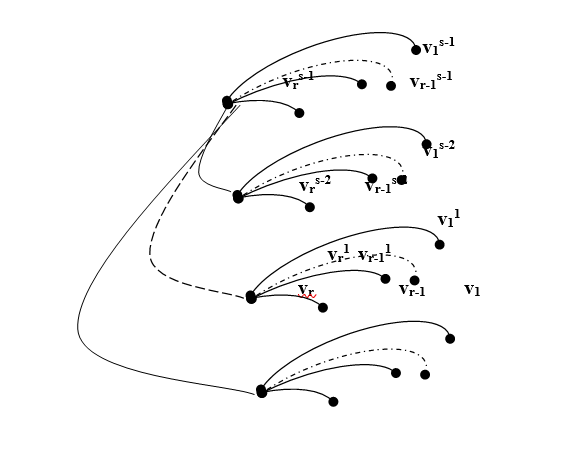
****

Fig. 5: Spanning tree

**Theorem 3.7:** The Planar Grid Graph Ps × Pr admits vertex odd power mean labeling.

**Proof:** The Planar Grid Graph Ps × Prhas 2sr + s + r vertices and 4sr edges.

**v1 v2 vr**

**v1ʹv2ʹvrʹ vr+1ʹ**

**v1s-2v2s-2vrs-2 vr+1s-2**

**v1s-1v2s-1vrs-1**

Fig. 6: Planar Grid Graph Ps × Pr

Define the vertex labeling f: V(Ps × Pr) →{1, 3, …, 2q +1} as

f(vi) = 2i – 1 for 1 ≤ i ≤ r

f(vi(1)) = f(vr) + 2i for 1 ≤ i ≤ r + 1

f(vi(2)) = f(vr(1)) + 2i for 1 ≤ i ≤ r

…….

f(vi(s-2)) = f(vr(s-3)) + 2i for 1 ≤ i ≤ r + 1

f(vi(s-1)) = f(vr(s-2)) + 2i for 1 ≤ i ≤ r

Thus the vertex labels are distinct and the Planar Grid Graph Ps × Pr satisfies the vertex odd power mean labeling condition. Therefore, the graph is a vertex odd power mean graph.

**Theorem 3.8:** *The graph Pr ʘ Ks admits vertex odd power mean labeling*.

**Proof:** The graph Prʘ Kshas *sr* vertices and *sr* – 1 edges.

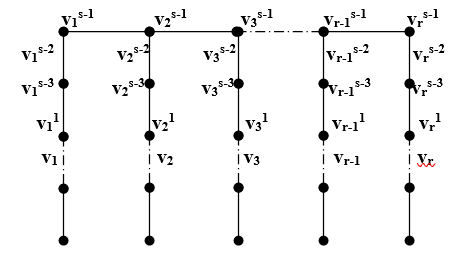
****

Fig. 7: Graph Pr ʘ Ks

Define the vertex labeling f: V(Pr ʘ Ks) →{1, 3, …, 2q +1} as

f(vi) = 2(s+1)i – 2s – 1 for 1 ≤ i ≤ r

f(vi(1)) = 2(s+1)i – 2s + 1 for 1 ≤ i ≤ r

f(vi(2)) = 2(s+1)i – 2s + 3 for 1 ≤ i ≤ r

…….

f(vi(s-2)) = 2(s+1)i - 3 for 1 ≤ i ≤ r

f(vi(s-1)) = 2(s+1)i - 1 for 1 ≤ i ≤ r

Therefore the vertex labels are distinct and the graph Pr ʘ Kssatisfies the vertex odd power mean labeling condition. As a result, the graph is a vertex odd power mean graph.

**Theorem 3.9:** *The Graph Pr + sSr**admits vertex odd power mean labeling*.

**Proof:** Thegraph Pr + sSr has *sr* vertices and *sr* - 1 edges.

Define the vertex labeling f: V(Pr + sSr) →{1, 3, …, 2q +1} as

f(vi) = 2i - 1 for 1 ≤ i ≤ r(s + 1) – 1

i ≠ r + 1, 2r + 2, …, (s + 1) (r – 1)

f(viʹ) = 2si – 1 for 1 ≤ i ≤ r

**v1ʹ v2ʹ vr-1ʹ vrʹ**

**v1  vr vr+2 v2r+1 vrs-1  vr(s+1)-1**

Fig. 8: Graph Pr + sSr

Here the vertex labels are distinct. So, the graph Pr + sSr is a vertex odd power mean graph.

# Conclusion

In this paper, the vertex odd power mean labeling is verified for the graphs such as the disconnected cycle graphs sCr , the Subdivision graphs *Ps,r and* S(Ks,r), *Firecracker graph, grid graph, Spanning Tree, the graphs Pr ʘ Ks, and Pr + sSr*

# References

1. Rosa. A (1967), On certain valuations of the vertices of a graph, Theory of Graphs (International Symposium, Rome, July 1966), Gordon and Breach, New York and Dunod Paris, pp 349-355.
2. Gallian. J. A (2022), A dynamic survey of graph labelling, *The Electronic Journal of Combinatorics*, pp 307 – 315.
3. Badr. E. M (2015), On graceful labeling of the generalization of cyclic snakes, *Journal of Discrete Mathematical Sciences and Cryptography*, 18(6), pp 773-783.
4. Hsu. D.F (1982), Harmonious labelings of Windmill Graphs and related graphs, *Journal of Graph Theory*, 6(1), pp 85–87.
5. Shendra Shainy. V and Balaji. V (2020), Even felicitous labeling, *Malaya Journal of Matematik*, S(1), pp 45-47.
6. Somasundaram. S and Ponraj. R (2003), Mean labelings of graphs, *National Academic Science and Letcures,* 26, pp 210-213.
7. P. Mercy and S. Somasundaram (2017), Power Mean Labeling of some Standard Graphs, Asia Pacific Journal of Research S.N.45797.
8. Kurugal Munikempanna Nagaraja, Sampathkumar Ramachandraiah and Venkataramana Bathahalli Siddappa (2021), Power Exponential Mean Labeling of Graphs, *Montes Taurus Journal of Pure and Applied Mathematics,* 3(2), pp 70 – 79.
9. Sreeji. S and Sandhya. S. S (2020), Power 3 Mean Labeling of Graphs, *International Journal of Mathematical Analysis,* 14(2), pp 51 – 59.
10. Haray. F (1988), Graph Theory, Narosa Publishing House Reading, New Delhi.
11. Vimala. C and Kavitha. B (2020), Vertex Odd Power Mean Labeling of graphs, *International Journal of Trend in Scientific Research and Development*.
12. Poovila. V and Vimala. C (2025), Application of vertex odd power mean labeling in Cryptography, *AIP Conference Proceedings,* 3252(020216).