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Contact Stiffness of a Rigid Flat-Ended Cylindrical Indenter on an Elastic Quarter-Space

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Abstract. The contact stiffness of an elastic quarter-space indented by a rigid cylindrical flat punch is numerically investigated using the Boundary Element Method. The simulations show that edge effects reduce contact stiffness, which gradually converges to the half-space solution as the indenter–edge distance increases. The results are further compared with the case of a side surface constrained not to move in the direction perpendicular to the edge. Although this boundary condition produces slightly higher contact pressures and stiffness when the contact region is near the edge, it provides a reasonable approximation of the fully free boundary while offering improved computational efficiency.

INTRODUCTION

The half-space is an idealized model widely used in contact mechanics [1], while edge effects are often unavoidable in practical applications. When the contact region lies on or near an edge, as in rail–wheel systems, rolling bearings, and gears, the edge effect should be considered [2-6]. The quarter-space contact problem specifically addresses the edge effects.

Hetényi proposed that the quarter-space problem can be solved by iteratively overlapping symmetric loads on two half-spaces [7]. Subsequently, Keer and his co-workers [8, 9] improved this method by employing a direct solution instead of the iterative process. In order to improve computational efficiency, many numerical methods have been introduced successively, such as the Ritz's method [10], the explicit solution [11], fast correction method [12, 13] and the fast Fourier transform algorithm [14, 15]. Nevertheless, when more complex phenomena such as adhesion are considered, a large number of mesh elements is required, and solving the quarter-space contact problem still demands a high computational cost.

Recently, Li and Popov [16] attempted to approximate the contact of quarter-space with free side surface by that with a freely sliding side surface. The quarter-space contact with a freely sliding side can be calculated equivalently as a symmetrically loaded contact on a half-space, substantially enhancing the computational efficiency.

This study considers a rigid cylindrical indenter pressed into an elastic quarter-space, to investigate the edge effects on contact stiffness. The elastic quarter-space is modeled with both free side and freely sliding side respectively, to assess the deviations introduced by this approximation.

METHOD

Figure 1 presents the schematic of the analytical model considered in this study. The top and side surfaces of the elastic quarter-space are defined by the planes $z = 0$ and $x = 0$, respectively, with the y-axis forming the only edge. A rigid cylindrical indenter is pressed on the top surface of the quarter-space. L is the distance between the center of the indenter and the edge, R is the radius of the cylindrical indenter, u is the deformation of top surface, and d is the indentation depth of the indenter. The parameters, u and d, are defined as positive in the positive z-axis direction. The contact on the elastic quarter-space with a free side or freely sliding side is illustrated in Figure 1(a) and (b), respectively.

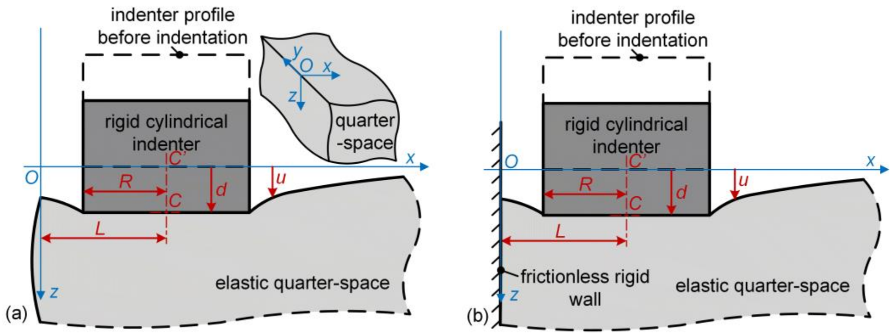

FIGURE 1. Schematic diagram of analytical models for the quarter-space contact with (a) a free side surface and (b) a freely sliding side surface. In the case of (b), the side surface of the quarter-space is constrained not to move in the direction perpendicular to the edge.

The method for solving the contact problems of a quarter-space is described in [17]. As shown in Figure 2(a), the stresses p_z acting on the top surfaces of a quarter-space are equivalent to the superposition of stresses p_A and p_B applied symmetrically on half-spaces A and B, respectively, (see Refs. [9, 17]). Here are the relationships

$$\begin{aligned} p_A &= (\mathbf{I} - \mathbf{K} \cdot \mathbf{K})^{-1} \cdot p_z, \\ p_B &= \mathbf{K} \cdot (\mathbf{K} \cdot \mathbf{K} - \mathbf{I})^{-1} \cdot p_z, \end{aligned} \tag{1}$$

where **K** is the influence matrix, and the method for calculating **K** can be found in the appendix of the reference [9].

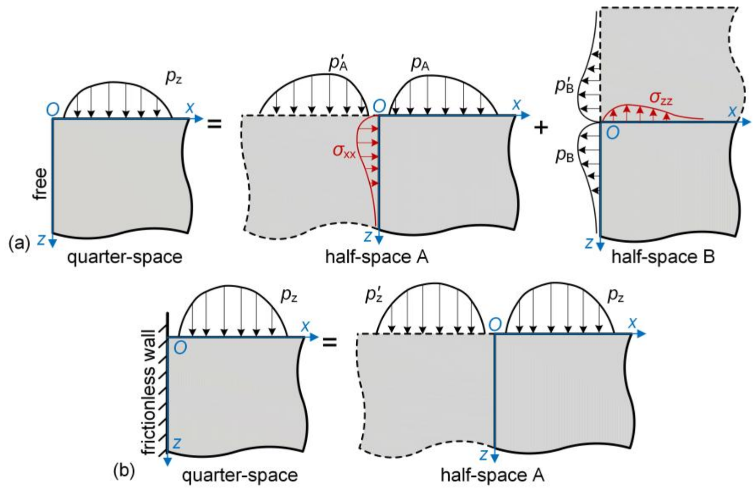

FIGURE 2. Computational methods for the quarter-space contact with (a) a free side or (b) a freely sliding side.

In the case of contact of quarter-space with a freely sliding side, the side surface of the quarter-space is constrained not to move in the direction perpendicular to the edge. This condition can be visualized as the side surface leaning against or adhering to a frictionless wall. This contact problem with load p_z on the top surface can be easily solved by applying an additional symmetric p_z, on an elastic half-space, as illustrated in Figure 2(b), which leads to a substantial reduction in computational cost.

In this way, the contact problem on a quarter-space with a free or freely sliding side is transformed into a contact problem on a half-space. Given an indentation depth, the deformation distribution within the contact region can be determined from the geometry of the flat indenter. The pressure distribution is then obtained using half-space theory, and the normal force is found by integration. The contact stiffness follows as the derivative of the normal force with respect to indentation depth. Since the relevant theories are well established [1], they are not detailed here.

RESULTS AND DISCUSSION

Figure 3 shows the variation of the contact stiffness k_n with the indenter–edge distance L. The quarter-space contact stiffness k_n is normalized with respect to the half-space solution

$$\bar{k}_n = k_n / k_{hs}\text{, with } k_{hs} = 2RE^*, \tag{2}$$

where k_{hs} is the contact stiffness of a cylindrical indenter on an elastic half-space with effective elastic modulus E^* which is given by

$$E^* = \frac{E}{1-\nu^2}, \tag{3}$$

where E is the elastic modulus and ν is the Poisson's ratio. The distance is normalized by the radius of the cylinder, $\bar{L} = L/R$.

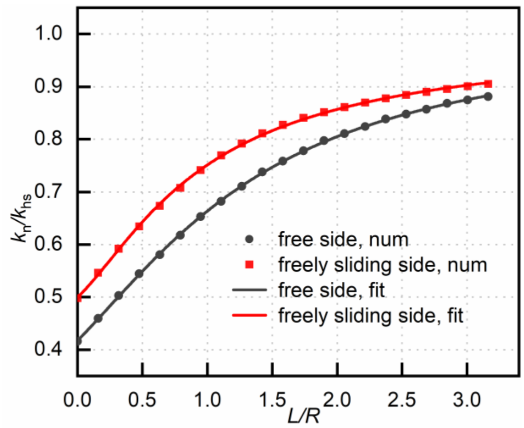

FIGURE 3. Dependence of contact stiffness on the distance between the indenter center and the edge of the elastic quarter-space. Markers are numerical results and fittings are presented with solid lines.

It can be observed from Figure 3 (solid circles and squares) that, as the distance increases, the contact stiffness on the quarter-space gradually increases and approaches the half-space solution. Compared with the free-side case, the quarter-space with a freely sliding side exhibits a higher contact stiffness. When the indenter center is directly located on the edge with $L/R = 0$, the quarter-space contact with a freely sliding side can be rigorously regarded as half of a half-space contact, resulting in a contact stiffness of 0.5.

The numerical results in Figure 3 can be fitted by the following expression

$$\bar{k}_n = \frac{\bar{L}^2 + 1.26463\bar{L} + 1.1494}{\bar{L}^2 + 1.39267\bar{L} + 2.75334} \text{ for free side case,} \tag{4}$$

$$\bar{k}_n = \frac{\bar{L}^2 + 0.64475\bar{L} + 0.40989}{\bar{L}^2 + 0.91571\bar{L} + 0.81978} \text{ for freely slide case.} \tag{5}$$

The fittings are presented with solid lines in Figure 3.

The contact pressure distributions on the quarter-space are shown in Figure 4. The pressure is normalized by the half-space solution of the pressure at the indenter center, p_{hs}. p_{hs} can be obtained by

$$p_{hs} = \frac{dE^*}{\pi R}. \tag{6}$$

From Figure 4, it can be observed that the edge effect reduces the contact pressure near the boundary, resulting in a left-right asymmetric pressure distribution. This asymmetry is further illustrated in Figure 5. Due to the finite mesh resolution, the contact pressure near the edge of the flat indenter remains finite. At L/R = 1.1, the pressure on the side of the indenter close to the edge (y-axis) of quarter-space is significantly lower than that on the opposite side.

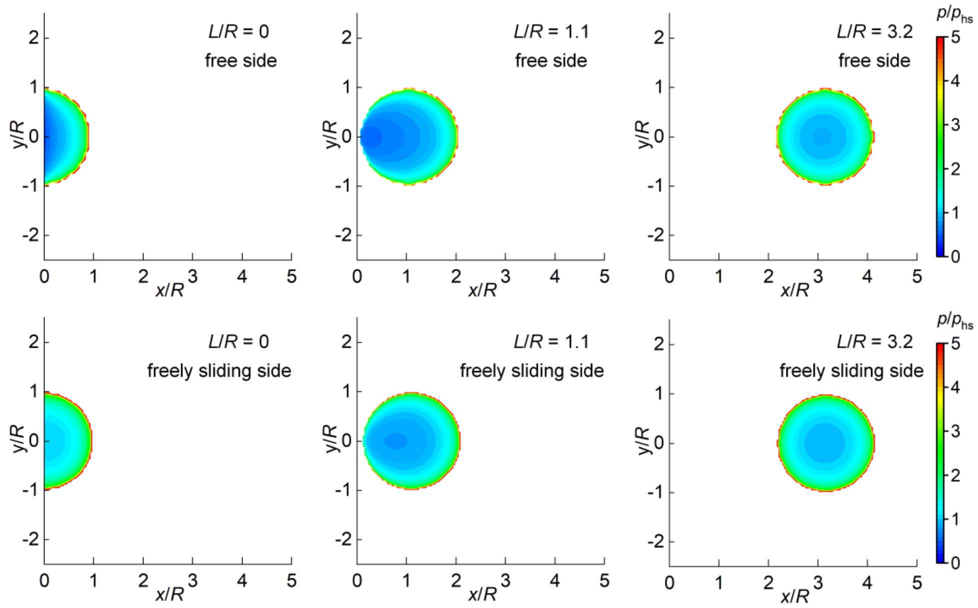

FIGURE 4. Contact pressure distribution on the top surface of the elastic quarter-space at different indenter locations.

As the distance increases, the edge effect gradually diminishes, and the pressure distribution approaches the symmetric half-space solution, as shown in Figures 4 and 5. Both the free-side and freely sliding-side cases exhibit this trend, although the quarter-space with a freely sliding side shows higher contact pressures near the edge.

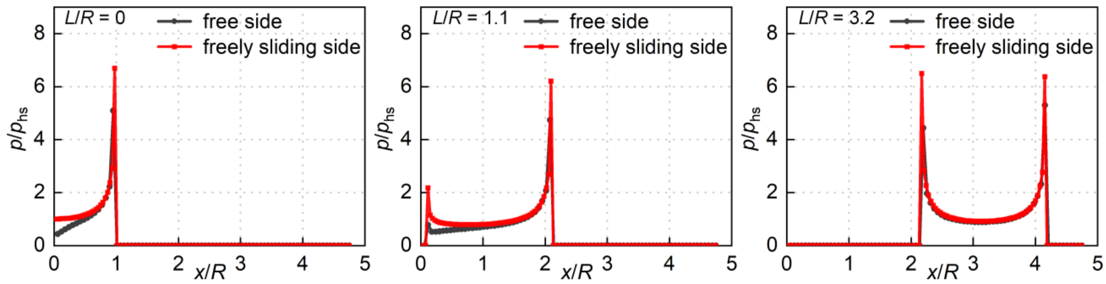

FIGURE 5. Pressure distribution along the centerline of the contact region (cross-section y = 0 in Figure 4).

CONCLUSION

The edge effect reduces the contact stiffness of the quarter-space, and this influence gradually vanishes as the indenter–edge distance increases. For both free and freely sliding sides, the contact pressure distribution and the variation trend of contact stiffness exhibit consistent behavior. The freely sliding side case can be regarded as a reasonable approximation of free side case, although it results in slightly higher contact pressures and contact stiffness when the contact region is close to the edge. The contact stiffness at different locations may have a strong influence on the adhesive contacts as well as the tangential contacts [18].

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