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Analytical Investigation of the Transient Dynamic Response of a Continuum with Discrete Inclusions

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Abstract. Scattering of non-stationary waves propagating in a continuum by inclusion is discussed. More precisely, the reflection of a plane pulse by a cylindrical void in an elastic medium is considered. An analytical approach presuming the construction of a non-stationary problem based on a solution for a harmonic wave is borrowed. In the initial approximation, a P-wave incident on the cylinder generates reflected P- and SV waves. The potentials of the incident and reflected waves give rise to a displacement and stress field in the continuum. In the reported study, the displacement field is considered. The solution obtained for a given frequency of the incident wave is generalized for a pane pulse by applying the Fourier integral. The displacement time history at a specified location is then monitored. The study appears as part of a numerical investigation into the transient wave process in a continuum due to the presence of scatterers, offering an alternative point of view to that based on finite-element analysis.

INTRODUCTION

A number of models have been proposed recently to assess the dynamic properties of composites based on the scattering properties of the inclusions. Liu et al. [1] developed a numerical Green's function-based approach to characterize the attenuation in two-phase matrix-inclusion microstructures. The authors compared the outcomes of the numerical and analytical solutions for various cases defined, assuming both phases are isotropic, with a constant density, and by varying density and elasticity. The authors concluded that numerical and analytical approaches are in good agreement if presuming differences in elastic properties only. Kulkarni et al. [2] investigated the propagation of ultrasonic waves in a polymer matrix composite with a dispersed phase of inclusions, using the finite element method. The effect of various factors (such as size, volume fraction of inclusions, and addition of interphase layer) on the attenuation characteristics of ultrasonic longitudinal waves in the matrix was investigated for harmonic waves by varying their frequency in the range of 1 - 4 MHz. Kamalinia et al. [3] presented a derivation of upwind numerical fluxes for the space discontinuous Galerkin finite-element method, for numerical modeling of wave propagation in multidimensional coupled acoustic/elastic media. They performed an eigenanalysis to highlight the eigenmodes of wave propagation and 'upwind' numerical fluxes on the interfaces (acoustic/acoustic and acoustic/elastic), in terms of exact solutions of relevant Riemann problems. Dorval et al. [4] conducted numerical simulations of longitudinal and shear waves' propagation through small representative elementary volumes to estimate velocity- and scattering-induced attenuation in an effective homogeneous material. Numerical results were compared to an established theoretical attenuation model. A model, recently proposed by Ru [5], defined an explicit framework for the dynamic behavior of fiber-reinforced unidirectional composites subjected to P-, SV-, and SH-waves. The approach was further refined by Basiri et al. [5] to accurately predict the dynamic behavior of composites at higher frequencies, near and beyond the bandgap region, by considering wave radiation damping. According to the Authors, the model is in good agreement with known numerical results presented in the literature.

To account for additional phenomena, i.e., to investigate the wave propagation in viscoelastic materials, the Volterra formalism has been implemented in the equations of elasodynamics [7, 8]. For this purpose, models employing fractional derivatives or fractional order operators have also been proposed [9, 10]. In some works [11-14], the linear viscoelastic behavior was determined by applying an integral Laplace transform in time, finding the solution in the space of images, and projecting it back (in the space of originals).

The investigation reported in this contribution complements a numerical analysis of the non-stationary wave process modification that results from the presence of scatterers in the continuum. The analytical modeling of the scattering of a pulse propagating in an elastic continuum by an inclusion within a representative cell (analogous to a representative volume element) provides an alternative standpoint to that one may have based on the output of a finite-element analysis. Apart from using the Laplace transform in time, the nonstationary solution can be constructed based on a solution for a specified frequency through the evaluation of its Fourier integral [15, 16]. This approach is illustrated below.

THE PLANE PULSE IN A REPRESENTATIVE VOLUME CELL

The problem of interest is the propagation of a plane pulse in an elastic medium containing inclusions, more precisely cylindrical voids, as shown in Fig. 1.

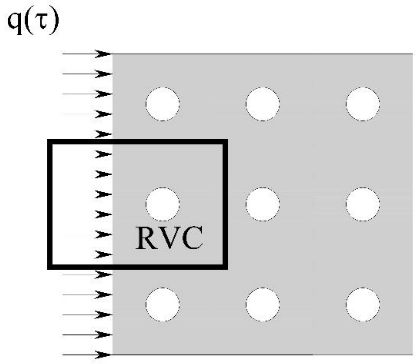

FIGURE 1. Elastic continuum with regularly distributed cylindrical voids.

where

$$q(\tau) = C \cdot \left(1 - e^{-50\tau}\right), \quad \tau > 0\,, \tag{1}$$

C is a constant and $\tau = t / t_0$, $t_0 = a / (2c_L)$ (a shown in Fig.2),

$$c_L = \sqrt{(\lambda + 2\mu) / \rho}\,, \tag{2}$$

λ, μ are the Lamé parameters,

$$\lambda = \frac{E \cdot \nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} \tag{3}$$

ρ is the material density, E is the elasticity modulus, and ν is the Poisson's ratio.

It can be solved by applying a numerical algorithm, for example, finite-element analysis. On the other hand, for a representative volume cell (RVC), the solution can be relatively easily obtained by using an analytical approach (with numerical implementation). Therefore, an auxiliary problem is defined by isolating the RVC (Fig. 2).

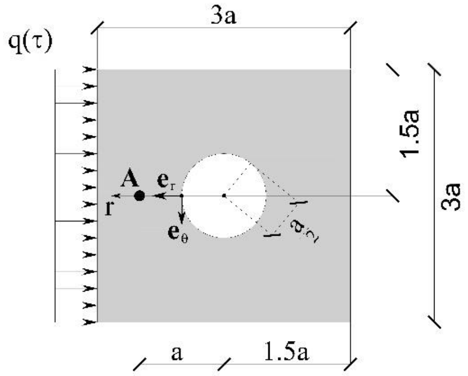

FIGURE 2. Representative volume cell for the continuum with inclusions.

In the auxiliary problem, an incident harmonic wave of amplitude A and frequency ω is be represented as follows:

$$P^{(i)} = A\sum_{n=0}^{\infty}\varepsilon_n \cdot i^n \cdot J_n\left(\alpha r\right)\cdot\cos\left(n\theta\right)\cdot e^{-i\omega t} \tag{4}$$

where J_n denotes the Bessel function of the first kind and $\alpha = \omega / c_L$. After having reached the cylindrical void, the incident P wave gives rise to reflected P- and SV waves:

$$P^{(r)} = \sum_{n=0}^{\infty} C_1 \cdot H_n^{(1)}\left(\alpha r\right)\cdot\cos\left(n\theta\right)\cdot e^{-i\omega t} \tag{5}$$

$$SV^{(r)} = \sum_{n=0}^{\infty} C_2 \cdot H_n^{(1)}\left(\beta r\right)\cdot\sin\left(n\theta\right)\cdot e^{-i\omega t} \tag{6}$$

In equations (5) and (6), $H_n^{(1)}$ denotes the Hankel function of the first kind and $\beta = \omega / c_t$, $c_t = \sqrt{\mu / \rho}$. The displacement field in the matrix can be defined by leveraging the potentials P^i (equation 2), P^r, *and* SV^r*:*

$$u_r = \frac{1}{r}\sum_{n=0}^{\infty}\left\{A\cdot\varepsilon_n\cdot i^n\cdot\Psi_1 + C_1\cdot\Psi_2 + C_2\cdot n\cdot H_n^{(1)}\left(\beta r\right)\right\}\cos\left(n\theta\right)\cdot e^{-i\omega t} \tag{7}$$

with

$$\Psi_1 = \alpha r J_{n-1}\left(\alpha r\right) - nJ_n\left(\alpha r\right), \tag{8}$$

$$\Psi_2 = \alpha r H_{n-1}\left(\alpha r\right) - nH_n\left(\alpha r\right). \tag{9}$$

The constants C_1 and C_2 are to be defined based on the boundary conditions on the interface between the scatterer and the matrix. Taking into account that for the void boundary, the free-surface conditions apply,

$$\sigma_{rr} = 0 \quad \text{and} \quad \sigma_{r\theta} = 0\,, \tag{10}$$

expressions for the corresponding stress components are required. The stress components are also defined by using the potentials defined in equations (7)-(9):

$$\sigma_{rr} = \frac{2\mu}{r^2}\sum_{n=0}^{\infty}\left\{A\cdot\varepsilon_n\cdot i^n\cdot F_{11}^{(1)}\left(\alpha r\right) + C_1 F_{11}^{(3)}\left(\alpha r\right) + C_2 F_{12}^{(3)}\left(\beta r\right)\right\}\cos\left(n\theta\right)\cdot e^{-i\omega t}\,, \tag{11}$$

$$\sigma_{r\theta} = \frac{2\mu}{r^2}\sum_{n=0}^{\infty}\left\{A\cdot\varepsilon_n\cdot i^n\cdot F_{41}^{(1)}\left(\alpha r\right) + C_1 F_{41}^{(3)}\left(\alpha r\right) + C_2 F_{42}^{(3)}\left(\beta r\right)\right\}\sin\left(n\theta\right)\cdot e^{-i\omega t}\,. \tag{12}$$

For the scattering boundary (i.e., $r = a/2$) one finds:

$$F_{11}^{(1)}\left(\alpha r\right) = \left(n^2 + n - \frac{1}{2}\beta^2\left(\frac{a}{2}\right)^2\right)J_n\left(\alpha\frac{a}{2}\right) - \alpha\frac{a}{2}J_{n-1}\left(\alpha\frac{a}{2}\right), \tag{13}$$

$$F_{11}^{(3)}\left(\alpha r\right)=\left(n^2+n-\frac{1}{2}\beta^2\left(\frac{a}{2}\right)^2\right)H_n^{(1)}\left(\alpha\frac{a}{2}\right)-\alpha\frac{a}{2}H_{n-1}^{(1)}\left(\alpha\frac{a}{2}\right), \quad (14)$$

$$F_{12}^{(3)}\left(\beta r\right)=-n\left(n+1\right)H_n^{(1)}\left(\beta\frac{a}{2}\right)+n\beta\frac{a}{2}H_{n-1}^{(1)}\left(\beta\frac{a}{2}\right), \quad (15)$$

$$F_{41}^{(1)}\left(\alpha r\right)=n\left(n+1\right)J_n\left(\alpha\frac{a}{2}\right)-n\alpha\frac{a}{2}J_{n-1}\left(\alpha\frac{a}{2}\right), \quad (16)$$

$$F_{41}^{(3)}\left(\alpha r\right)=n\left(n+1\right)H_n^{(1)}\left(\alpha\frac{a}{2}\right)-n\alpha\frac{a}{2}H_{n-1}^{(1)}\left(\alpha\frac{a}{2}\right), \quad (17)$$

$$F_{42}^{(3)}\left(\beta r\right)=-\left(n^2+n-\frac{1}{2}\beta^2\left(\frac{a}{2}\right)^2\right)H_n^{(1)}\left(\beta\frac{a}{2}\right)+\beta\frac{a}{2}H_{n-1}^{(1)}\left(\beta\frac{a}{2}\right), \quad (18)$$

From equations (10)-(12), with (13)-(18), the constants C_1 and C_2 are defined:

$$C_1=\frac{N_1}{D},\quad C_2=\frac{N_2}{D}, \quad (19)$$

$$N_1=-A\cdot\varepsilon_n\cdot i^n\left[F_{11}^{(1)}\left(\alpha\frac{a}{2}\right)F_{42}^{(3)}\left(\beta\frac{a}{2}\right)-F_{12}^{(3)}\left(\beta\frac{a}{2}\right)F_{41}^{(1)}\left(\alpha\frac{a}{2}\right)\right], \quad (20)$$

$$N_2=-A\cdot\varepsilon_n\cdot i^n\left[F_{11}^{(3)}\left(\alpha\frac{a}{2}\right)F_{41}^{(1)}\left(\alpha\frac{a}{2}\right)-F_{11}^{(1)}\left(\alpha\frac{a}{2}\right)F_{41}^{(3)}\left(\alpha\frac{a}{2}\right)\right], \quad (21)$$

$$D=F_{11}^{(3)}\left(\alpha\frac{a}{2}\right)F_{42}^{(3)}\left(\beta\frac{a}{2}\right)-F_{12}^{(3)}\left(\beta\frac{a}{2}\right)F_{41}^{(3)}\left(\alpha\frac{a}{2}\right), \quad (22)$$

Thus, the radial component of the displacement takes the form:

$$u_r=\frac{1}{r}\sum_{n=0}^{\infty}A\cdot\varepsilon_n\cdot i^n\cdot\left[\frac{\Psi_1 D-\Omega_1\Psi_2-\Omega_2\cdot n\cdot H_n^{(1)}\left(\beta r\right)}{D}\right]\cos\left(n\theta\right)\cdot e^{-i\omega t}, \quad (23)$$

$$\Omega_1=F_{11}^{(1)}\left(\alpha\frac{a}{2}\right)F_{42}^{(3)}\left(\beta\frac{a}{2}\right)-F_{12}^{(3)}\left(\beta\frac{a}{2}\right)F_{41}^{(1)}\left(\alpha\frac{a}{2}\right), \quad (24)$$

$$\Omega_2=F_{11}^{(3)}\left(\alpha\frac{a}{2}\right)F_{41}^{(1)}\left(\alpha\frac{a}{2}\right)-F_{11}^{(1)}\left(\alpha\frac{a}{2}\right)F_{41}^{(3)}\left(\alpha\frac{a}{2}\right). \quad (25)$$

The above solution obtained for a specific value of ω can then be converted into a solution for a plane pulse as follows:

$$u_r=\frac{1}{4\pi c_1}\sum_{n=0}^{\infty}\varepsilon_n\cdot i^{n+1}\cdot\cos\left(n\theta\right)\cdot\int_{-\infty}^{\infty}\frac{1}{z}e^{z}\cdot U^*\cdot e^{-iz\tau}dz, \quad (26)$$

where

$$U^*=\frac{\Psi_1 D-\Omega_1\Psi_2-\Omega_2\cdot n\cdot H_n^{(1)}\left(\beta r\right)}{D}. \quad (27)$$

The results are displayed in Fig. 3. For the elastic homogeneous medium, a model material is employed with elasticity modulus $E=210000MPa$, Poisson's ratio $\nu=0.3$, and density $\rho=7850kg/m^3$.

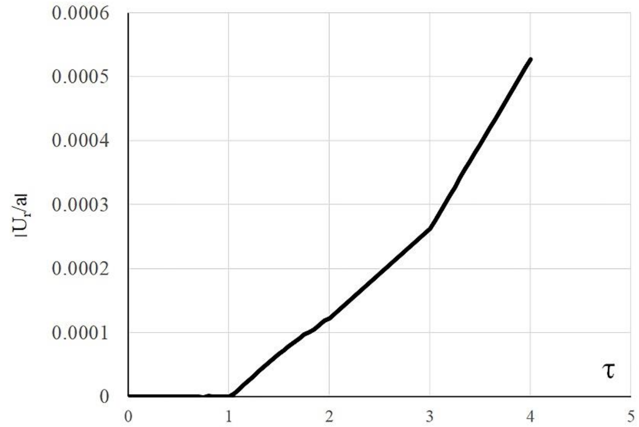

FIGURE 3. Radial displacement at point A (as defined in Fig. 2).

The displacement time history is obtained for $\Delta\tau = 0.01$. At location A, the wave arrives at $\tau = 1$ (given that the disturbance propagates with a speed c_L); at $\tau = 2$, the incident pulse reaches the void, and at $\tau = 3$, the displacement field is obtained as a superposition of components resulting from the incident disturbance and an effect due to the scattering by the void.

CONCLUSION

An analytical approach aimed at determining the non-stationary wave process in an elastic continuum generated by a plane pulse has been discussed. Specifically, a disturbance provoked by a plane pulse, as well as its scattering by an inclusion (void) of a predefined form, has been considered.

The approach prescribes to construct the non-stationary solution via a Fourier integral, leveraging the fields of interest (i.e., displacement and stress fields) obtained for the potentials associated with harmonic incident and scattered waves. As an illustration, the displacement evolution at a specified location has been obtained.

Within the forthcoming research works, the analytical approach will be enriched to take into consideration viscoelasticity and plasticity. A comparison with results obtained by finite element analysis is also presumed.

REFERENCES

1. F. Liu, A. P. Argüelles, C. Peco, A Green's function-based method for wave attenuation on random matrix-inclusion microstructures with local isotropy. Computer Methods in Applied Mechanics and Engineering **446**, 118334 (2025). https://doi.org/10.1016/j.cma.2025.118334
2. S. S. Kulkarni, A. Tabarraei, P. P. Ghag, "A finite element approach for study of wave attenuation characteristics of epoxy polymer composite," in ASME International Mechanical Engineering Congress and Exposition 52149 (American Society of Mechanical Engineers 2018, November), pp. V009T12A042
3. H. Kamalinia, A. Barbarulo, B. Tie, A coupled acoustic/elastic discontinuous Galerkin finite element method: Application to ultrasonic imaging of 3D-printed synthetic materials. Computers & Structures **291**, 107208 (2024). https://doi.org/10.1016/j.compstruc.2023.107208
4. V. Dorval, N. Leymarie, A. Imperiale, E. Demaldent, P. E. Lhuillier, Numerical estimation of ultrasonic phase velocity and attenuation for longitudinal and shear waves in polycrystalline materials. Ultrasonics **148**, 107517.7 (2025). https://doi.org/10.1016/j.ultras.2024.107517
5. C. Q. Ru, A direct method for wave propagation in elastic fiber composites. Mathematics and Mechanics of Solids **28**(10), 2242-2255 (2023). https://doi.org/10.1177/10812865231158589
6. A. Basiri, C. Q. Ru, P. Schiavone, A refined analytical model for acoustic waves in elastic fiber composites. Mathematics and Mechanics of Solids, 10812865251336256 (2025). https://doi.org/10.1177/10812865251336256

7. Y. N. Rabotnov, *Elements of Hereditary Solid Mechanics* (Nauka Publishers, Moscow, 1977). [in Russian]
8. M. H. Ilyasov, Dynamical torsion of viscoelastic cone. TWMS J. Pure Appl. Math **2**(2), 203-220 (2011).
9. Y. A. Rossikhin, M. V. Shitikova, and P. T. Trung, Analysis of the Viscoelastic Sphere Impact against a Viscoelastic Uflyand-Mindlin Plate considering the Extension of Its Middle Surface. Shock and Vibration, **2017**(1), 5652023 (2017). https://doi.org/10.1155/2017/5652023
10. F. Mainardi, *Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models* (World Scientific, 2022).
11. L. A. Igumnov, E. A. Korovaytseva, and S. G. Pshenichnov, (2021) "Dynamics, Strength of Materials and Durability in Multiscale Mechanics," in *Advanced Structured Materials-2021,* (Springer Nature, Switzerland AG, 2021), 137, pp. 89-96.
12. I. Colombaro, A. Giusti, F. Mainardi, On transient waves in linear viscoelasticity. Wave Motion **74**, 191-212 (2017). https://doi.org/10.1016/j.wavemoti.2017.07.008
13. R. Christensen, *Theory of viscoelasticity: an introduction* (Elsevier, 2012).
14. J. D. Achenbach, Vibrations of a viscoelastic body. AIAA journal **5**(6), 1213-1214 (1967). https://doi.org/10.2514/3.4173
15. S. K. Kanaun, V. M. Levin, and F. J. Sabina, Propagation of elastic waves in composites with random set of spherical inclusions (effective medium approach). Wave motion **40**(1), 69-88 (2004). https://doi.org/10.1016/j.wavemoti.2003.12.013
16. A. C. Eringen, E. S. Suhubi, *Elastodynamics, vol. II* (Academic Press, New York, 1975).