A Problem of Relaxation Filtering of Suspensions With Forming a Cake Layer

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Abstract. This article derives filtration equations for the formation of a relaxing layer based on the laws of suspension conservation. The equations are solved numerically, utilizing the Stefan problem and the moving front capture method to model the layer's growth. The numerical results are then used to quantify how relaxation phenomena influence the material's filtration properties.

INTRODUCTION

Filtration of suspensions through porous media is of great practical importance. The processes of formation of a precipitate layer on the filter surface have been studied in the literature [1,2,3,4,5]. If the liquid (dispersion medium) surrounding the solid particles in the suspension is a polymer solution or a very dense liquid, then the suspension itself may exhibit non-Newtonian (i.e., viscosity is pressure-dependent) properties [6]. On a filter, suspensions exhibit properties characteristic of a porous medium. Crucially, the formed cake layer is not purely viscous; it displays viscoelastic behavior. While one could link filtration models to the rheological models of relaxing suspensions, it's more convenient to use relaxation filtration laws. This approach posits that the suspension's inherent relaxation properties are the direct source of the relaxation phenomena observed during filtration [7, 8, 9]. The relationship between the pressure gradient ∇p and the filtration velocity $\vec{v}$ defined by Darcy's Law often shows inconsistencies between experimental and theoretical results. This discrepancy suggests that $\vec{v}$ isn't simply proportional to ∇p under all conditions. The failure of the simple linear relationship is likely due to the liquid possessing non-equilibrium rheological properties (like viscoelasticity), as well as complex interactions within the porous medium. These factors include: (1) The interaction of the liquid with the porous matrix. (2) The adsorption of oil components onto the solid surfaces. (3) Filling and emptying of pores by large molecules, such as those in polymeric liquids [10]. These effects violate the law's underlying equilibrium nature, causing it to acquire a relaxing character [11-14]. Researchers have explored various approaches to generalize Darcy's law [15, 16]. For instance, a memory model for fluid movement in porous media was proposed, which successfully matched experimental flow velocities for water moving through sand [17]. Furthermore, another study introduced a memory formalism using fractional derivatives into the constitutive equations that govern pressure-flow and pressure-density relationships [18]. Collectively, experimental finding across this work is that memory effects cause a delay in flow velocity, with the maximum steady state only being reached at later times.

In this paper aims to modify Darcy's law using established filtration theories to develop the governing equations for suspension filtration with cake formation. Since the cake layer thickness increases over time, the moving boundary between the suspension and the sediment must be tracked. This requires an additional equation, which leads to a Stefan-type differential problem. Consequently, the resulting system of equations is solved numerically. The primary steps involve formulating the appropriate mathematical model, obtaining the numerical solution, and then analyzing the results to describe key process parameters. Under pressure, a suspension is driven toward a porous medium (the filter). Since the solid particles are unable to pass through, they are completely retained on the upstream side, where they accumulate to form a filter cake. The liquid component, or filtrate, is separated and flows through both the cake and the filter medium. As filtration progresses, the thickness of the cake, $L(t)$, continuously increases over time.

EQUATIONS OF CAKE FILTRATION

Let us suppose, what the filtration velocity of the liquid phase relative to the pressure gradient has a nonequilibrium nature. The nonequilibrium relationship is assumed to be in linear differential form

$$q_\ell = -\frac{k}{\mu}\left(1+\lambda_{p\ell}\frac{\partial}{\partial t}\right)\frac{\partial p_\ell}{\partial x}, \qquad (1)$$

where q_ℓ - liquid phase velocity, k - permeability coefficient, μ - viscosity, p_ℓ - pressure in the liquid phase, $\lambda_{p\ell}$ - relaxation time of filtration velocities, t - time, x - distance away from the medium.

Given that the phase filtration rate is scale-dependent, relaxation events also occur within distinct time frames. This allows us to simplify the model by neglecting the relaxation contribution of the solid phase to the filtration rate, as it is considered minor compared to the effect exerted by the liquid phase [19, 20].

Using equation (1), a governing equation for p_s (compressive stress) is derived using conservation laws:

$$\frac{\partial p_s}{\partial t} = \frac{k^0 p_A}{\mu\beta}\left(1+\frac{p_s}{p_A}\right)^{1-\beta}\frac{\partial}{\partial x}\left[\left(1+\frac{p_s}{p_A}\right)^{\beta-\delta}\left(1+\lambda_{p\ell}\frac{\partial}{\partial t}\right)\left(\frac{\partial p_s}{\partial x}\right)\right] - q_{\ell m}\frac{\partial p_s}{\partial x}, \qquad (2)$$

where p_A - characteristic pressure, k^0 - value of k at $p_s = 0$, β, δ - constants.

The flow rate $q_{\ell m}$ is

$$q_{\ell m} = \frac{k}{\mu}\left(1+\lambda_{p\ell}\frac{\partial}{\partial t}\right)\frac{\partial p_s}{\partial x}\bigg|_{x=0}. \qquad (3)$$

The conservation law is expressed by the equation [3]

$$\frac{\partial(q_\ell + q_s)}{\partial x} = 0.$$

This means that in a given speed regime, the total flow rate ($q_\ell + q_s = const$) is constant. This is in sharp contrast to a system operating under a given pressure regime, where the total pressure ($p_\ell + p_s$) is not constant but rather a function of time, $p_\ell + p_s = r(t)$, which must be determined while solving the problem.

For the purposes of this analysis, we will focus on the given speed regime, where the total flow rate is constant:

$$q_\ell + q_s = v_0 = const.$$

The initial and boundary conditions for equation (2) will then be set according to this regime.

$$p_s(0,x) = 0,\ \frac{k}{\mu}\left(1+\lambda_{p\ell}\frac{\partial}{\partial t}\right)\frac{\partial p_s}{\partial x}\bigg|_{x=0} = -\frac{p_\ell}{\mu R_m}\bigg|_{x=0} = -v_0 = const < 0,\ p_s(t, L(t)) = 0, \qquad (4)$$

where R_m - relative resistance of the filtering element.

The equation of thickness growth for the cake layer $L(t)$ has the form

$$\frac{dL}{dt} = -\frac{\varepsilon_s^0}{\varepsilon_s^0 - \varepsilon_{s_0}}\left[\frac{k}{\mu}\left(1+\lambda_{p\ell}\frac{\partial}{\partial t}\right)\frac{\partial p_s}{\partial x}\right]\Bigg|_{L^-} + \left[\frac{k}{\mu}\left(1+\lambda_{p\ell}\frac{\partial}{\partial t}\right)\frac{\partial p_s}{\partial x}\right]\Bigg|_{x=0}, \qquad (5)$$

where ε_s^0 - solid content at zero pressure, ε_{s_0} - concentration of solid particles in suspension.

From equation (5) we can determine a mobile front $L(t)$ - the boundary between the suspension and the cake layer. This equation is solved together with the basic filtering equation (2) under the conditions (4) and $L(0) = 0$.

We introduce the following notations

$$a\left(p_s\right)=\left(1+\frac{p_s}{p_A}\right)^{1-\beta},\ b\left(p_s\right)=\left(1+\frac{p_s}{p_A}\right)^{\beta-\delta},\ c\left(p_s\right)=\frac{k^0}{\mu}\left(1+\frac{p_s}{p_A}\right)^{-\delta},\ c^0\left(p_s\right)=\frac{k^0}{\mu}\left(1+\frac{p_s}{p_A}\right)^{-\delta}\Bigg|_{x=0},\ c_1=\frac{k^0 p_A}{\beta\mu},$$

$$c_2=\frac{\varepsilon_s^0}{\varepsilon_s^0-\varepsilon_{s_0}}.$$

With taking into account these notations equation (2) can be transformed into the following form

$$\frac{\partial p_s}{\partial t}=c_1 a\left(p_s\right)\frac{\partial}{\partial x}\left[b\left(p_s\right)\left(1+\lambda_{p\ell}\frac{\partial}{\partial t}\right)\left(\frac{\partial p_s}{\partial x}\right)\right]-q_{\ell m}\frac{\partial p_s}{\partial x}. \tag{6}$$

The equation for the mobile boundary $L(t)$, (5), takes the form

$$\frac{dL}{dt}=-c_2\left[c\left(p_s\right)\left(1+\lambda_{p\ell}\frac{\partial}{\partial t}\right)\frac{\partial p_s}{\partial x}\right]\Bigg|_{L^-}+q_{\ell m}, \tag{7}$$

where

$$q_{\ell m}=c^0\left(p_s\right)\left[\left(1+\lambda_{p\ell}\frac{\partial}{\partial t}\right)\frac{\partial p_s}{\partial x}\right]\Bigg|_{x=0}.$$

To solve the problem (6), (7) with (4) and $L(0)=0$ we use the finite differences method [21, 22].

We introduce a uniform grid by t with the step τ $\bar{\omega}_\tau=\left\{t\,|\,t=t_j=j\tau,\ j=0,1,...,N,\ \tau N=T\right\}$, and a non - uniform grid by coordinate x [21, 22] $\bar{\omega}_h=\left\{x\,|\,x=x_i=x_{i-1}+h_i,\ h_i=0,\ i=1,2,...,N,N+1,N+1,...,\ x_N=L\right\}$ with the variable steps $h_i>0$. We are to choose the steps h_i from the interval $[x_i, x_{i+1}]$ so, that the mobile boundary moves exactly on one step along the time grid. This approach is known as the method of catching the front in a grid node. We denote by $p_{s,i}^{j+1}$ the grid function corresponding to p_s. We approximate equation (6) by an implicit difference scheme that is nonlinear with respect to the function $p_{s,i}^{j+1}$

$$\frac{p_{s,i}^{j+1}-p_{s,i}^{j}}{\tau}=c_1\frac{2a\left(p_{s,i}^{j}\right)}{h_i+h_{i+1}}\left\{b(p_{s,i+1/2}^{j+1})\frac{p_{s,i+1}^{j+1}-p_{s,i-1}^{j+1}}{h_i+h_{i+1}}+\frac{\lambda_{pl}}{\tau}b(p_{s,i+1/2}^{j+1})\left[\frac{p_{s,i+1}^{j+1}-p_{s,i-1}^{j+1}}{h_i+h_{i+1}}-\frac{p_{s,i+1}^{j}-p_{s,i-1}^{j}}{h_i+h_{i+1}}\right]-b(p_{s,i-1/2}^{j+1})\frac{p_{s,i}^{j+1}-p_{s,i-1}^{j+1}}{h_i}-\right.$$

$$\left.-\frac{\lambda_{pl}}{\tau}b(p_{s,i-1/2}^{j+1})\left[\frac{p_{s,i}^{j+1}-p_{,si-1}^{j+1}}{h_i}-\frac{p_{s,i}^{j}-p_{s,i-1}^{j}}{h_i}\right]\right\}-\left(q_{\ell m}\right)_0^{j+1}\frac{p_{s,i}^{j+1}-p_{s,i-1}^{j+1}}{h_i},\ i=1,...,N-1,\quad j=0,1,...,N-1, \tag{8}$$

where

$$a\left(p_{s,i}^{j}\right)=\left(1+\frac{p_{s,i}^{j}}{p_A}\right)^{1-\beta},\ b\left(p_{s,i+1/2}^{j+1}\right)=\frac{1}{2}\left[\left(1+\frac{p_{s,i+1}^{j+1}}{p_A}\right)^{\beta-\delta}+\left(1+\frac{p_{s,i}^{j+1}}{p_A}\right)^{\beta-\delta}\right],\ c^0\left(p_{s,0}^{j+1}\right)=\frac{k^0}{\mu}\left(1+\frac{p_{s,0}^{j+1}}{p_A}\right)^{-\delta},$$

$$\left(q_{\ell m}\right)_0^{j+1}=c^0\left(p_{s,0}^{j+1}\right)\left(\frac{p_{s,1}^{j+1}-p_{s,0}^{j+1}}{h_0}+\frac{\lambda_{p\ell}}{\tau}\left(\frac{p_{s,1}^{j+1}-p_{s,0}^{j+1}}{h_0}-\frac{p_{s,1}^{j}-p_{s,0}^{j}}{h_0}\right)\right).$$

In Eq. (7), $\frac{dL}{dt}\approx\frac{h_{i+1}}{\tau}$ after the approximation can be written in the form

$$\frac{h_{i+1}}{\tau}=-c_2\left[c\left(p_{s,i-1/2}^{j}\right)\left(\frac{p_{s,i}^{j+1}-p_{s,i-1}^{j+1}}{h_{i+1}}+\frac{\lambda_{p\ell}}{\tau}\left(\frac{p_{s,i}^{j+1}-p_{s,i-1}^{j+1}}{h_{i+1}}-\frac{p_{s,i}^{j}-p_{s,i-1}^{j}}{h_{i+1}}\right)\right)\right]+\left(q_{\ell m}\right)_0^{j+1}, \tag{9}$$

where

$$c\left(p_{s,i-1/2}^{j}\right)=\frac{k^0}{2\mu}\left[\left(1+\frac{p_{s,i}^{j}}{p_A}\right)^{-\delta}+\left(1+\frac{p_{s,i-1}^{j}}{p_A}\right)^{-\delta}\right].$$

Approximation of initial and boundary conditions (4) gives

$$p_{s,i}^{j}=0\,,\ i=0,1,..,N\,,\ j=0\,,\ -\mu c^{0}\left(p_{s,0}^{j}\right)\left(\frac{p_{s,1}^{j+1}-p_{s,0}^{j+1}}{h_1}+\frac{1}{\tau}\left(\frac{p_{s,1}^{j+1}-p_{s,0}^{j+1}}{h_1}-\frac{p_{s,1}^{j}-p_{s,0}^{j}}{h_1}\right)\right)=\frac{p_{\ell}}{R_m}=v_0\,,\ j=\overline{0,N}\,,\ p_{s,i}^{j+1}=0\,,$$

$$i=N+1,N+2,...,\quad j=0,1,....\tag{10}$$

The obtained set of equations (8) is nonlinear, so to solve it we use the method of simple iteration

$$\frac{\overset{(\sigma+1)}{p_{s,i}^{j+1}}-\varphi_i^j}{\tau}=c_1\frac{2a\left(p_{s,i}^{j}\right)}{h_i+h_{i+1}}\left\{b(\overset{(\sigma)}{p_{s,i+1/2}^{j+1}})\frac{\overset{(\sigma+1)}{p_{s,i+1}^{j+1}}-\overset{(\sigma+1)}{p_{s,i-1}^{j+1}}}{h_i+h_{i+1}}+\frac{\lambda_{pl}}{\tau}b(\overset{(\sigma)}{p_{s,i+1/2}^{j+1}})\left[\frac{\overset{\sigma+1)}{p_{s,i+1}^{j+1}}-\overset{(\sigma+1)}{p_{s,i-1}^{j+1}}}{h_i+h_{i+1}}-\frac{p_{s,i+1}^{j}-p_{s,i-1}^{j}}{h_i+h_{i+1}}\right]-b(\overset{(s)}{p_{s,i-1/2}^{j+1}})\frac{\overset{(\sigma+1)}{p_{s,i}^{j+1}}-\overset{(\sigma+1)}{p_{s,i-1}^{j+1}}}{h_i}-\right.$$

$$\left.-\frac{\lambda_{pl}}{\tau}b(\overset{(\sigma)}{p_{s,i-1/2}^{j+1}})\left[\frac{\overset{(\sigma+1)}{p_{s,i}^{j+1}}-\overset{(\sigma+1)}{p_{,si-1}^{j+1}}}{h_i}-\frac{p_{s,i}^{j}-p_{s,i-1}^{j}}{h_i}\right]\right\}-\left(\overset{(\sigma)}{q_{\ell m}}\right)_0^{j+1}\frac{\overset{(\sigma+1)}{p_{s,i}^{j+1}}-\overset{(\sigma+1)}{p_{s,i-1}^{j+1}}}{h_i},\tag{11}$$

where

$$b\left(\overset{(\sigma)}{p_{s,i+1/2}^{j+1}}\right)=\frac{1}{2}\left[\left(1+\frac{\overset{(\sigma)}{p_{s,i+1}^{j+1}}}{p_A}\right)^{\beta-\delta}+\left(1+\frac{\overset{(\sigma)}{p_{s,i}^{j+1}}}{p_A}\right)^{\beta-\delta}\right],\ \left(\overset{(\sigma)}{q_{\ell m}}\right)_0^{j+1}=c^0\left(\overset{(\sigma)}{p_{s,0}^{j+1}}\right)\left(\frac{\overset{(\sigma)}{p_{s,1}^{j+1}}-\overset{(\sigma)}{p_{s,0}^{j+1}}}{h_1}+\frac{\lambda_{p\ell}}{\tau}\left(\frac{\overset{(\sigma)}{p_{s,1}^{j+1}}-\overset{(\sigma)}{p_{s,0}^{j+1}}}{h_1}-\frac{p_{s,1}^{j}-p_{s,0}^{j}}{h_1}\right)\right),$$

σ is the number of iteration.

It can be seen that the system of equations (10) is now linear with respect to $\overset{(s+1)}{p}{}_{s,i}^{j+1}$, which allows us to use the Tomas's algorithm [21]. As a condition to stop iteration procedure on this time layer, the following relationship can be used:

$$\max_i\left|\overset{(s+1)}{p_{s,i}^{j+1}}-\overset{(s)}{p_{s,i}^{j+1}}\right|\le\varepsilon\,,\tag{12}$$

where ε is the given accuracy of calculation.

When condition (10) is satisfied then $\overset{(s+1)}{p}{}_{s,i}^{j+1}=p_{s,i}^{j+1}$. As an initial approach we can use $\overset{(s=0)}{p_{s,i}^{j+1}}=p_{s,i}^{j}$. Equation (11) leads to the system of linear equations

$$\overset{(\sigma)}{A_i}\overset{(\sigma+1)}{p_{s,i-1}^{j+1}}-\overset{(\sigma)}{B_i}\overset{(\sigma+1)}{p_{s,i}^{j+1}}+\overset{(\sigma)}{C_i}\overset{(\sigma+1)}{p_{s,i+1}^{j+1}}=-\overset{(\sigma)}{F_i}\,,\ i=\overline{1,N-1}\,,\tag{13}$$

where

$$\overset{(\sigma)}{A_i}=-\frac{1}{h_i+h_{i+1}}\left(1+\frac{\lambda_{p\ell}}{\tau}\right)b\left(\overset{(\sigma)}{p_{s,i+1/2}^{j+1}}\right)+\frac{1}{h_i}\left(1+\frac{\lambda_{p\ell}}{\tau}\right)b\left(\overset{(\sigma)}{p_{s,i-1/2}^{j+1}}\right)+\frac{h_i+h_{i+1}}{2c_1h_ia\left(p_{s,i}^{j}\right)}q_{\ell m}\,,$$

$$\overset{(\sigma)}{B_i}=\frac{1}{h_i}\left(1+\frac{\lambda_{p\ell}}{\tau}\right)b\left(\overset{(\sigma)}{p_{s,i-1/2}^{j+1}}\right)+\frac{h_i+h_{i+1}}{2\tau c_1a\left(p_{s,i}^{j}\right)}+\frac{h_i+h_{i+1}}{2c_1h_ia\left(p_{s,i}^{j}\right)}q_{\ell m}\,,\ \overset{(\sigma)}{C_i}=\frac{1}{h_i+h_{i+1}}\left(1+\frac{\lambda_{p\ell}}{\tau}\right)b\left(\overset{(\sigma)}{p_{s,i+1/2}^{j+1}}\right),$$

$$\overset{(\sigma)}{F_i}=\frac{h_i+h_{i+1}}{2\tau c_1a\left(p_{s,i}^{j}\right)}p_{s,i}^{j}+\frac{\lambda_{p\ell}}{\left(h_i+h_{i+1}\right)\tau}b\left(\overset{(\sigma)}{p_{s,i+1/2}^{j+1}}\right)\left(p_{s,i-1}^{j}-p_{s,i+1}^{j}\right)-\frac{\lambda_{p\ell}}{h_i\tau}b\left(\overset{(\sigma)}{p_{s,i-1/2}^{j+1}}\right)\left(p_{s,i-1}^{j}-p_{s,i}^{j}\right).$$

The equation (9) is used to determine the step h_{i+1} and it can be written in the form

$$\left(h_{i+1}\right)^2-\tau\left(q_{\ell m}\right)_0^{j+1}h_{i+1}+\tau c_2\,c\left(p_{s,i-1/2}^{j}\right)\left(p_{s,i}^{j+1}-p_{s,i-1}^{j+1}+\frac{\lambda_{p\ell}}{\tau}\left(p_{s,i}^{j+1}-p_{s,i-1}^{j+1}-p_{s,i}^{j}+p_{s,i-1}^{j}\right)\right)=0.$$

By solving this nonlinear equation for each temporal layer we can determine h_{i+1}. The system of linear algebraic equations (13) is solved by the Tomas' algorithm

$$\overset{(\sigma+1)}{p_{s,i}^{j+1}}=\xi_{i+1}\overset{(\sigma+1)}{p_{s,i+1}^{j+1}}+\zeta_{i+1}\,,\tag{14}$$

where

$$\xi_{i+1}=\frac{\overset{(\sigma)}{C_i}}{\overset{(\sigma)}{B_i}-\overset{(\sigma)}{A_i}\xi_i},\ \zeta_{i+1}=\frac{\overset{(\sigma)}{F_i}+\overset{(\sigma)}{A_i}\zeta_i}{\overset{(\sigma)}{B_i}-\overset{(\sigma)}{A_i}\xi_i}.$$

The starting values of the coefficients ξ_1 and ζ_1 are determined from the boundary condition (10), which have the form

$$\xi_1=1,\ \zeta_1=\frac{c^0\left(p_{s,0}^j\right)\frac{\lambda_{p\ell}}{h_0\tau}\left(p_{s,0}^j-p_{s,1}^j\right)+v_0}{\frac{c^0\left(p_{s,0}^j\right)}{h_0}\left(1+\frac{\lambda_{p\ell}}{\tau}\right)}. \quad (15)$$

NUMERICAL RESULTS AND DISCUSSION

Numerical results with using (14), (15) were obtained with the following values of initial parameters: $v_0=10^{-4}$ m/s, $p_A=10^4$ Pa, $R_m=10^{12}$ 1/m, $\mu=10^3$ Pa·s, $k^0=0,8\cdot10^{-13}$ m², $\varepsilon_s^0=0.20$, $\varepsilon_{s_0}=0,0076$, $\beta=0,13$, $\delta=0,57$.

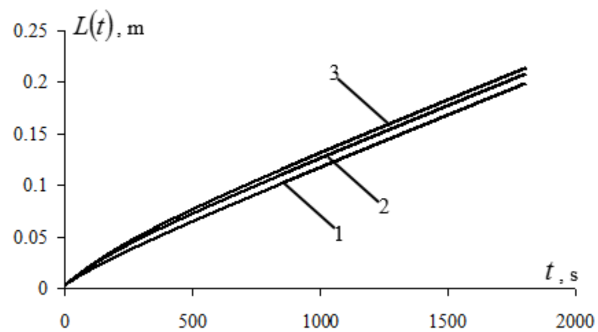

FIGURE 1. Dynamics of the cake thickness at $\lambda_{p\ell}=0$ (1); 150 (2); 350 (3) c.

Figure 1 graphically illustrates the dynamics of cake thickness, $L(t)$, over time, t, for three different values of the relaxation time, $\lambda_{p\ell}$:

- Curve 1: $\lambda_{p\ell}=0$ (no relaxation)
- Curve 2: $\lambda_{p\ell}=150$
- Curve 3: $\lambda_{p\ell}=350$

The numerical results show that an increase in the relaxation time, $\lambda_{p\ell}$, leads to a faster growth rate of the cake thickness at all other constant filtration conditions.

CONCLUSION

The results show that the relaxation properties of the flow have a significant effect on both the growth of the cake thickness and the overall filtration properties. In particular, the relaxation effects in the filtration laws reduce the distribution of the compressive pressure (p_s) and the liquid pressure (p_ℓ) inside the cake. However, when the process is sufficiently advanced, i.e., when the flow time significantly exceeds the characteristic relaxation time, the effect of relaxation phenomena ceases (or disappears).

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