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Study of Non-Stationary Processes in Inhomogeneous Viscoelastic Materials

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Abstract. Initial boundary value problems that describe transient wave processes in inhomogeneous linearly viscoelastic bodies are investigated. A piecewise homogeneous structure with continuity conditions at the contact of homogeneous components is chosen as the main type of heterogeneity. At the same time, for viscoelastic functionally graded materials, their continuous heterogeneity is approximated by a layered homogeneous medium. The hereditary properties of homogeneous viscoelastic components are characterized by linear integral Boltzmann-Volterra constitutive relations with relaxation kernels of various types. In the mathematical formulation of the initial boundary value problem for a piecewise homogeneous viscoelastic body, the perturbations propagation area is considered limited. To construct solutions, the integral Laplace transform in time and the operation of its reversal are used. The solution in the originals is written in a new form. Special relations are presented that establish a certain correspondence between kernels belonging to different classes of functions. Using the example of a non-stationary problem for a functionally graded spherical layer, the similarity of non-stationary processes is demonstrated when using various singular kernels connected by these special relations.

Key words: inhomogeneous viscoelastic materials, hereditary kernels, Laplace transform, non-stationary waves, viscoelastic dynamic problems.

INTRODUCTION

The development of modern technologies, as well as the emergence of new materials, lead to the necessity of studying the dynamics of heterogeneous media with hereditary properties. These can be piecewise homogeneous structures with viscoelastic components, as well as viscoelastic functionally graded materials (FGM) with continuous spatial heterogeneity. Some results in this area, obtained recently by various authors for piecewise homogeneous materials, are contained, for example, in works [1-5]. At the same time, it should be noted that in most well–known publications, attention is paid not to transient wave processes, but to harmonic vibrations and waves [6-8]. In the field of viscoelastic FGM dynamics, other authors' research is devoted to stationary processes (for example, [9, 10]). Only some characteristic publications on the topic are mentioned here. Among the various approaches to studying non-stationary waves in media of this type, the development of analytical and numerical analytical methods is very important, but today the successes achieved in this area are clearly insufficient. This is especially true for transient wave processes in piecewise homogeneous viscoelastic bodies with a large number of homogeneous components and in FGM.

This paper contains the results of research in the field of unsteady dynamics of viscoelastic inhomogeneous media within the framework of the Boltzmann-Volterra model, conducted by using numerical analytical methods based on previous theoretical developments [11 – 13]. The solutions of initial boundary value problems of viscoelasticity for piecewise homogeneous bodies are presented in a special form that is suitable for different types of relaxation kernels.

When using linear integral Boltzmann-Volterra constitutive relations, the question arises as to how belonging of viscoelastic kernels to one or another class of functions affects the transition process. This issue has already been raised earlier [11, 14, 15], and here the proposed approach has been further developed. Using the example of a non-stationary problem for a spherical FGM layer, the similarity of non-stationary processes is demonstrated when using various singular kernels in the Boltzmann-Volterra model, connected by special relations.

FORMULATION OF THE NON-STATIONARY PROBLEM. THE SOLUTION REPRESENTATION FORM

The layered homogeneous medium with continuity conditions at the contact of the layers is considered the main type of heterogeneity. The continuous heterogeneity of viscoelastic FGMs will be approximated by a structure consisting of a large number of viscoelastic homogeneous layers. This approach is widely used in modeling stationary processes in elastic, thermoelastic, and piezoelectroelastic FGMs [16 – 18]. In non-stationary problems, the validity of such an approximation is confirmed by calculations for various bodies (see, for example, [19]). Let us consider a nonstationary linear viscoelasticity problem for the N-component piecewise homogeneous body, including the equations of dynamics ($n = 1, 2, ..., N$):

$$(\hat{\lambda}_n + \hat{\mu}_n)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}^{(n)}(\mathbf{x},t) + \hat{\mu}_n \Delta \mathbf{u}^{(n)}(\mathbf{x},t) + \mathbf{f}^{(n)}(\mathbf{x},t) = \rho_n \ddot{\mathbf{u}}^{(n)}(\mathbf{x},t), \quad \mathbf{x} \in \Omega_n \,, \qquad (1)$$

constitutive relations:

$$\tilde{\boldsymbol{\sigma}}^{(n)}(\mathbf{x},t) = 2\hat{\mu}_n\, \tilde{\boldsymbol{\varepsilon}}^{(n)}(\mathbf{x},t) + \hat{\lambda}_n \mathrm{div}\, \mathbf{u}^{(n)}(\mathbf{x},t)\, \tilde{\mathbf{I}}, \quad \tilde{\boldsymbol{\varepsilon}}^{(n)} = \mathrm{def}\, \mathbf{u}^{(n)}, \quad \mathbf{x} \in \Omega_n \,, \qquad (2)$$

boundary conditions (m is the number of the component that has common points with the boundary Σ):

$$\tilde{\boldsymbol{\alpha}}^{(m)}(\mathbf{x})\, \tilde{\boldsymbol{\sigma}}^{(m)}(\mathbf{x},t)\mathbf{n} + \tilde{\boldsymbol{\beta}}^{(m)}(\mathbf{x})\, \mathbf{u}^{(m)}(\mathbf{x},t) = \mathbf{p}^{(m)}(\mathbf{x},t), \quad \mathbf{x} \in \Sigma, \quad t > 0 \,, \qquad (3)$$

continuity conditions at the contact of adjacent components with numbers p and q:

$$\mathbf{u}^{(p)}(\mathbf{x},t) = \mathbf{u}^{(q)}(\mathbf{x},t), \quad \tilde{\boldsymbol{\sigma}}^{(p)}(\mathbf{x},t)\,\mathbf{n} = \tilde{\boldsymbol{\sigma}}^{(q)}(\mathbf{x},t)\mathbf{n}, \quad \mathbf{x} \in \Sigma_{pq} \,, \qquad (4)$$

and initial conditions:

$$\mathbf{u}^{(n)}(\mathbf{x},0) = \mathbf{b}_1^{(n)}(\mathbf{x}), \quad \dot{\mathbf{u}}(\mathbf{x},0) = \mathbf{b}_2^{(n)}(\mathbf{x}), \quad \mathbf{x} \in \Omega_n \,, \qquad (5)$$

herewith the operators $\hat{\lambda}_n$ and $\hat{\mu}_n$ have the form

$$\hat{\lambda}_n = \frac{1}{3}[3K_0^{(n)}(1 - \hat{T}_{\mathrm{v}}^{(n)}) - 2G_0^{(n)}(1 - \hat{T}_{\mathrm{s}}^{(n)})], \quad \hat{\mu}_n = G_0^{(n)}(1 - \hat{T}_{\mathrm{s}}^{(n)}), \quad \hat{T}_j^{(n)} \xi(t) = \int_0^t T_j^{(n)}(t-\tau)\xi(\tau)\,d\tau, \quad j = \mathrm{v,s} \qquad (6)$$

Here the indexes of the values correspond to the components numbers; $\tilde{\boldsymbol{\sigma}}^{(n)}$, $\tilde{\boldsymbol{\varepsilon}}^{(n)}$ are the stress and small strain tensors; $\mathbf{u}^{(n)}$ is the displacement vector; n is an exterior unit normal to the corresponding boundary; ρ_n is the density; Δ is Laplace operator; $\tilde{I}$ is the unit tensor; $G_0^{(n)}$, $K_0^{(n)}$ are the instantaneous values of the shear and bulk moduli; $T_{\mathrm{v}}^{(n)}(t)$, $T_{\mathrm{s}}^{(n)}(t)$ are the volumetric and shear relaxation kernels; $\tilde{\boldsymbol{\alpha}}^{(m)}$, $\tilde{\boldsymbol{\beta}}^{(m)}$ are given second-rank tensors that determine the type of boundary conditions; $\mathbf{p}^{(m)}$, $\mathbf{f}^{(n)}$, $\mathbf{b}_1^{(n)}$, $\mathbf{b}_2^{(n)}$ are given vectors of boundary effects, volumetric forces, initial displacements and velocities; the dot above the variable denotes the derivative with respect to time t. The disturbances propagations domain Ω consists from N subdomains: $\Omega = \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_N$, which correspond to homogeneous isotropic components of the body, herewith Ω and its boundary Σ are limited. This does not necessarily mean that the body itself is limited. The dimension of the coordinate vector $\mathbf{x}$ depends on the specific problem and can vary from 1 to 3.

The paper [11] contains the results of general theoretical studies of problem (1)–(6) based on the application of the integral Laplace transform with respect to time and the subsequent construction of a solution in the originals. It was established the relationship between the properties of the set E of eigenvalues of the free oscillations problem for the body under consideration and the branching points of the solution of the non–stationary problem (1) - (6) in the space of Laplace images, as well as the relationship between the elements of the set E and the poles of the solution in the images. In [12], new forms of the solution to the linear viscoelasticity problem for a homogeneous body are presented. For the case of a piecewise homogeneous viscoelastic medium considered here, similar mathematical considerations make it possible to write down the solution in a new form, taking into account the set of homogeneous components of the material.

So, let us apply the integral Laplace transform in time to the problem (1) – (6). Denote the images of the functions $\mathbf{u}^{(n)}(\mathbf{x},t)$, $\tilde{\boldsymbol{\sigma}}^{(n)}(\mathbf{x},t)$, $\mathbf{p}^{(m)}$, $\mathbf{f}^{(n)}$, $T_{\mathrm{v}}^{(n)}(t)$, $T_{\mathrm{s}}^{(n)}(t)$ as $\mathbf{U}^{(n)}(\mathbf{x},s)$, $\tilde{\mathbf{S}}^{(n)}(\mathbf{x},s)$, $\mathbf{P}^{(m)}$, $\mathbf{F}^{(n)}$, $\Theta_{\mathrm{v}}^{(n)}(s)$, $\Theta_{\mathrm{s}}^{(n)}(s)$ $(s \in C)$ respectively.

Assume that there are no volumetric forces: $\mathbf{f}^{(n)} \equiv \mathbf{0}$, and for surface loads the following relation is satisfied:

$$\mathbf{p}^{(n)}(\mathbf{x},t) = \mathbf{q}^{(n)}(\mathbf{x})\psi(t), \quad \psi(t) = h(t),$$

where $h(t)$ is the Heaviside function, and there are no displacements of the body as a rigid whole. Let herewith one of the following two cases be realized for the hereditary properties of a piecewise homogeneous body.

Case 1. All non-identically zero hereditary kernels $T_{\mathrm{v}}^{(n)}, T_{\mathrm{s}}^{(n)}$ belong to the class of functions that are partial sums of the Prony series:

$$\sum_{m=1}^{M} a_m \exp(-b_m t), \quad \text{where} \quad \sum_{m=1}^{M} a_m / b_m < 1, \quad a_m > 0, \quad b_m > 0 \quad (m = 1,2,...M), \tag{7}$$

Case 2. $T_{\mathrm{v}}^{(n)} \equiv 0$ for every n, and all non-identically zero $T_{\mathrm{s}}^{(n)}$ are Rzhanitsyn-Koltunov kernels (here the time is assumed to be dimensionless):

$$Ae^{-Bt}t^{\delta-1}, \quad \text{where} \quad A\Gamma(\delta)/B^{\delta} < 1, \quad A > 0, \quad B > 0, \quad 0 < \delta < 1, \tag{8}$$

$\Gamma(\delta)$ is a gamma-function with an appropriate argument. The constants a_m, b_m and M, as well as the constants A, B, δ for each component of a piecewise homogeneous medium, are, generally speaking, unique.

Besides, let the asymptotic behavior of the images $\mathbf{U}^{(n)}$ with $s \to \infty$ be "favorable" [12]. Then, reasoning similarly to how it was done in the work [12], we can represent the originals $\mathbf{u}^{(n)}$ in a form containing a series of residues at the complex poles z_j of the images, as well as two integrals (one over a finite segment, the other is improper):

$$\mathbf{u}^{(n)}(\mathbf{x},t) = \operatorname*{Res}_{s=0}[\mathbf{U}^{(n)}(\mathbf{x},s)e^{st}] + 2\sum_{j=1}^{\infty} \mathrm{Re}\{\operatorname*{Res}_{s=z_j}[\mathbf{U}^{(n)}(\mathbf{x},s)e^{st}]\} + \frac{1}{\pi}e^{\alpha_0 t}\int_0^{\omega_0} \mathrm{Re}\{\mathbf{U}^{(n)}(\mathbf{x},\alpha_0 + i\omega)e^{i\omega t}\}d\omega -$$

$$-\frac{1}{\pi}\int_{-\infty}^{\alpha_0} \mathrm{Im}\{\mathbf{U}^{(n)}(\mathbf{x},\alpha + i\omega_0)e^{i\omega_0 t}\}e^{\alpha t}d\alpha, \quad n = 1,2,...,N. \tag{9}$$

Here $z_j = \alpha_j + i\omega_j$, $\alpha_j, \omega_j \in R$, $\omega_j > 0$, $j = 1,2,3,....$, herewith $\alpha_j < 0$. The values α_0, ω_0 are chosen arbitrarily at intervals $\alpha_0 \in (s^*; 0)$, $\omega_0 \in (0; \omega_1)$, where $\omega_1 = \mathrm{Im}(z_1)$, z_1-is the complex pole with the smallest positive imaginary part, s^* is the largest among the roots of all equations:

$$1 - \Theta^{(n)} = 0, \quad 1 - \Theta_{\mathrm{s}}^{(n)} = 0, \tag{10}$$

where $\Theta_{\mathrm{s}}^{(n)}(s)$ are Laplace images of kernels $T_{\mathrm{s}}^{(n)}(t)$, and $\Theta^{(n)}$ are images of functions:

$$T^{(n)}(t) = [(1+\nu_0^{(n)})T_{\mathrm{v}}^{(n)}(t) + 2(1-2\nu_0^{(n)})T_{\mathrm{s}}^{(n)}(t)]\big/[3(1-\nu_0^{(n)})].$$

Note that $\{\alpha_j \pm i\omega_j\} \subset E$ and if a piecewise homogeneous body has at least one viscoelastic component, then the set E can have finite limit points on the complex plane. These points do not belong to E and are the roots of at least one of the equations (10). In the above two cases concerning the hereditary properties of the components, the roots of equations (10) are real negative ones.

The expression for the stress tensor $\tilde{\boldsymbol{\sigma}}^{(n)}$ through its image $\tilde{\mathbf{S}}^{(n)}$ is similar to (9). For other loading functions $\psi(t)$ solutions are obtained based on formula (9) and the Duhamel integral. The field of applicability of formula (9) is probably much broader than the indicated cases 1 and 2, but this issue requires further study.

SPECIAL RELATIONS BETWEEN RELAXATION KERNELS

When studying wave processes in linearly viscoelastic media, an important question arises about the choice of a class of functions for relaxation kernels $T_{\mathrm{v}}^{(n)}, T_{\mathrm{s}}^{(n)}$, which are included in the constitutive Boltzmann-Volterra relations (6). We introduce dimensionless hereditary kernels and time:

$$\gamma_s^{(n)}(\tau) = t_0 T_s^{(n)}, \quad \gamma_v^{(n)}(\tau) = t_0 T_v^{(n)}, \quad \tau = t / t_0, \tag{11}$$

where t_0 is some characteristic process time. Next, we will consider the dimensionless kernels $\gamma_s^{(n)}, \gamma_v^{(n)}$ in the class of functions (7) or (8), considering all constants and the time dimensionless in accordance with relations (11) and formally replacing t with τ in (7), (8). Note that the kernels from class (8) have a weak singularity at zero, unlike regular kernels (7). Consider the question of how significantly the belonging of relaxation kernels to class (7) or class (8) affects the nature of unsteady wave processes in a viscoelastic medium. In this regard, special relations have been proposed [11, 14, 15] that establish a certain correspondence between hereditary kernels from different classes of functions. Let us formulate these relations here using dimensionless quantities.

Let given a kernel $\gamma_1(\tau)$ (volumetric or shear relaxation) and chosen a class of kernels Q, defined by a known function $\Phi(\beta_1, \beta_2, \ldots, \beta_K, \tau)$, which depends on K the numerical parameters $(\beta_1, \beta_2, ..., \beta_K) \in \Omega_\beta \subset \mathbb{R}^K$ and the time τ. We construct the kernel $\gamma_2(\tau) = \Phi(\beta_1^0, \beta_2^0, \ldots, \beta_K^0, \tau) \in Q$ by determining the values $\beta_1^0, \beta_2^0, .., \beta_K^0$ from the condition of equality of the long-term moduli corresponding to the kernels γ_1 and γ_2:

$$\int_0^\infty \Phi(\beta_1, \beta_2, \ldots, \beta_K, \tau) d\tau = \int_0^\infty \gamma_1(\tau) dt, \tag{12}$$

and minimizing the integral (if it exists) by the parameters β_j:

$$\int_0^\infty [\Phi(\beta_1, \beta_2, \ldots, \beta_K, \tau) - \gamma_1(\tau)]^2 d\tau \xrightarrow{\beta_1, \beta_2, \ldots \beta_K} \min, \tag{13}$$

It was hypothesized that the kernels $\gamma_1(\tau)$ and $\gamma_2(\tau)$, connected by relations (12), (13), affect transient processes in viscoelastic materials similarly, all other initial data being equal, and some possible restrictions on the domain Ω_β of variation of the parameters β_j.

Consider as a class Q the set of functions of the form (7) with one term of the sum ($M = 1$), i.e., with only two parameters: $\beta_1 = a_1$, $\beta_2 = b_1$. It was proved [15] that if the initial kernel $\gamma_1(\tau)$ is a smooth function and $[\gamma_1(\tau)]^2$ is integrable on $(0;\infty)$, and the following conditions are satisfied:

$$\lim_{\tau \to +\infty} \tau\gamma_1(\tau) = 0, \quad \lim_{\tau \to 0+} \tau\gamma_1(\tau) = 0, \quad d\gamma_1(\tau)/d\tau < 0 \quad (\tau > 0), \tag{14}$$

then for $\gamma_1(\tau)$ there exists, moreover, a single kernel $\gamma_2(\tau) \in Q$:

$$\gamma_2(\tau) = \beta_1^0 \exp(-\beta_2^0 \tau), \tag{15}$$

satisfying conditions (12), (13). Herewith $\beta_1^0 = \beta_2^0\, \Theta_1(0)$, and β_2^0 is the only positive root of the algebraic equation (with respect to the unknown β):

$$\Theta_1(0) - 4[\Theta_1(\beta) + \beta\Theta_1'(\beta)] = 0, \quad \beta > 0, \tag{16}$$

where $\Theta_1(s)$ is the Laplace transform of $\gamma_1(\tau)$; the prime denotes the derivative. Note that if the initial kernel γ_1 is chosen from the set of functions (7) with $M \geq 2$, or (8) with the additional requirement $0.5 < \delta < 1$, then its square is integrable on $(0;\infty)$ and it has all the properties (14).

The hypothesis about the similarity of the influence of the corresponding hereditary kernels was confirmed by calculations for a number of specific examples [11, 14, 15], where the initial kernel $\gamma_1(\tau)$ was chosen either in the form (7) for $M \geq 2$, or in the form (8) for $0.5 < \delta < 1$, and the kernel $\gamma_2(\tau)$ was searched for in the two-parameter form (15). If the singularity parameter for the initial kernel γ_1 was chosen not too close to 0.5, namely: $0.6 \leq \delta < 1$, then in all examples the similarity of non-stationary processes was observed for the kernels $\gamma_1(\tau)$ and $\gamma_2(\tau)$. This similarity was assessed visually by graphs of the dependence of stresses and displacements on the time at various fixed points of the bodies. The dynamics of some homogeneous bodies [14, 15], as well as the dynamics of an infinite two-layer cylinder [11], were investigated. In the present work, relying on previous results, we consider the influence of singular relaxation kernels of the form (8) on transient processes in viscoelastic FGM. Herewith the parameter δ is varied, and the singular kernels themselves are connected by certain relations.

So, let the singular relaxation kernel γ_1 have the form (8). We construct a two-parameter kernel γ_2 (15) for it using relations (12), (13). Taking into account the equation (16), as well as the type of the kernel γ_1 and its image $\Theta_1(s)$, we obtain the following relations for the quantities β_1^0, β_2^0 :

$$\beta_2^0 = B[y_0(\eta)-1], \quad \beta_1^0 = \beta_2^0 A\Gamma(\delta)/B^{\delta}, \tag{17}$$

where $\eta = 1-\delta, \quad 0<\eta<0.5$, herewith $y_0(\eta)$ is the root of an algebraic equation (with a parameter η) relatively unknown y :

$$y^{2-\kappa} - 4\eta y - 4(1-\eta) = 0 .$$

For a given η ($0<\eta<0.5$), this root exists and is the only one, and $y_0(\eta) > 1$. The second relation (17) means the equality of the long-term moduli (12) corresponding to the kernels γ_1 and γ_2, and the first is obtained by minimizing the integral (13) by two parameters: β_1 and β_2.

Let us rewrite the relations (17) for the three parameters of the kernel γ_1 in the form

$$B = \beta_2^0/[y_0(\eta)-1], \quad A = \beta_1^0 B^{\delta}/[\beta_2^0\Gamma(\delta)]. \tag{18}$$

Thus, a regular two-parameter kernel γ_2 of the form (15) corresponds to an infinite set of singular three-parameter kernels γ_1 of the form (8). Choosing, for example, a certain value of the singularity parameter δ, one can easily find the remaining two parameters A, B of the singular kernel (8), which is in accordance with the kernel of the form (15) according to the relations (12), (13). Next, using a specific example of a spherical FGM layer, we will demonstrate the effect of hereditary kernels of the form (8) with various singularity parameters on transient wave processes. In this case, such singular kernels are in accordance with the same regular two-parameter kernel (15) according to the relations (18).

CALCULATION RESULTS FOR A SPHERICAL FGM LAYER

In a spherical coordinate system R, θ, ϕ, let us consider the problem of the unsteady longitudinal waves propagation in a spherical layer (hollow sphere) consisting of a viscoelastic FGM. The instantaneous values of the shear modulus G_0 and Poisson's ratio ν_0, as well as the density ρ of this material, generally speaking, continuously depend on the radial coordinate R. In addition, we will consider the kernel of the shear relaxation T_{s} to be a function of R and t, and the kernel of the volumetric relaxation T_{v} to be identically zero. Let's denote the inner and outer radii of the layer by R_0 and R_{sph}, accordingly. The initial state of the body is rest, but from the moment $t=0$ a uniformly distributed normal load $P(t)$ begins to act on its outer surface $R = R_{\mathrm{sph}}$, while the inner surface $R = R_0$ remains free (Fig. 1).

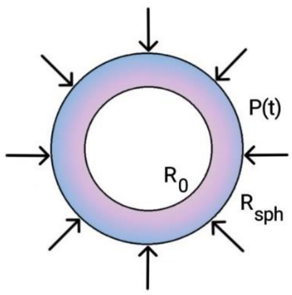

FIGURE 1. The central section of the FGM sphere and the loading scheme

We approximate the continuous heterogeneity of the FGM by concentric homogeneous viscoelastic spherical layers with kinematic and dynamic continuity conditions at the boundaries between them. The relations $R_{n-1} \le R \le R_n, \quad n = 1,2,3,...,N$ correspond to the layers, herewith $R_N = R_{\mathrm{sph}}, \quad N >> 1$. Let us introduce dimensionless values:

$$r = R / R_N, \quad r_0 = R_0 / R_N, \quad r_n = R_n / R_N \quad (n = 1,2,3,...,N, \quad r_N = 1), \quad \tau = t / t_0, \quad \psi(\tau) = P / F_0 \tag{19}$$

where $t_0 = R_N / c_N$, c_N is the longitudinal elastic wave velocity in the N -th layer; F_0 is the characteristic constant of the stress dimension. Dimensionless shear relaxation kernel $\gamma_s^{(n)}(\tau)$ has the form (8) taking into account the relations (11), in which it is accepted: $t_0 = R_N / c_N$. For the initial FGM layer also: $r_0 = R_0 / R_{sph} = R_0 / R_N$ and $\tau = t / t_0$, where $t_0 = R_{sph} / c(1)$, $c(r)$ is the longitudinal elastic wave velocity in the FGM. Note that the equality $c(1) = c_N$ is satisfied exactly or approximately, depending on the method of approximating the continuous heterogeneity by layers. The mathematical formulation of the spherically symmetric problem under consideration for a multilayer viscoelastic sphere is contained in [13]. This work also contains the solution to this problem in images after applying the integral Laplace transform in time and the solution in the originals based on formula (9).

Here we consider a spherical viscoelastic FGM layer with $r_0 = 0.8$ and the following initial data ($r_0 \le r \le 1$):

$$\frac{G_0(r)}{G_0(r_0)} = \frac{\rho(r)}{\rho(r_0)} = 1 + 4\frac{r - r_0}{1 - r_0}, \quad \nu_0 \equiv 0.3; \quad \gamma_v \equiv 0, \quad \gamma_s(r,\tau) = a(r)e^{-\tau}, \quad a(r) = 0.3(1-r)/(1-r_0). \tag{20}$$

It follows from relations (20) that the velocity of longitudinal elastic waves c does not depend on r, which is convenient when analyzing transients. The kernel γ_s has a form similar to (15), but with the coefficient $\beta_1^0 = a(r)$ that is not a constant, but a function that decreases linearly with respect to r. The values G_0, ρ increase linearly over r, herewith $G_0(1)/G_0(r_0) = \rho(1)/\rho(r_0) = 5$. We choose the function characterizing the load in the form of a smoothed step: $\psi(\tau) = 1 - e^{-50\tau}$ ($\tau > 0$) and then the load P limit with $\tau \to \infty$ is equal to F_0 (19). During the research were found the values of relative stresses:

$$\kappa_1(r,\tau) = \sigma_R / F_0, \quad \kappa_2(r,\tau) = \sigma_\theta / F_0,$$

where σ_R, σ_θ are radial and circumferential stresses ($\sigma_\theta = \sigma_\phi$). They were approximated by the corresponding values for a layered homogeneous structure:

$$\kappa_1^{(n)}(r,\tau) = \sigma_R^{(n)} / F_0, \quad \kappa_2^{(n)}(r,\tau) = \sigma_\theta^{(n)} / F_0, \quad n = 1,2,3,...,N,$$

herewith, the parameters of the FGM were approximated by 80 layers of equal thickness (this turned out to be quite sufficient). All calculations were carried out based on the solution of the problem for the corresponding multilayer hollow sphere [13] using formula (9).

For the given shear relaxation kernel (20), we introduce the notation $\gamma_0 = \gamma_s(r,\tau)$ and, for this kernel, according to relations (18), we find the parameters A, B of the Rzhanitsyn-Koltunov singular kernel (8) by specifying different values of the singularity parameter δ. Here, B is a constant, and A is a linearly decreasing function of the variable r. We denote the singular shear relaxation kernels constructed in this way for three different values of the parameter δ by $\gamma_1, \gamma_2, \gamma_3$ and present the expressions obtained for them:

$$\delta = 0.9 \quad (\eta = 0.1), \quad \gamma_1(r,\tau) = A(r)e^{-0.832\tau}\tau^{-0.1}, \quad A(r) = 0.238(1-r)/(1-r_0),$$

$$\delta = 0.7 \quad (\eta = 0.3), \quad \gamma_2(r,\tau) = A(r)e^{-0.508\tau}\tau^{-0.3}, \quad A(r) = 0.144(1-r)/(1-r_0),$$

$$\delta = 0.55 \quad (\eta = 0.45), \quad \gamma_3(r,\tau) = A(r)e^{-0.285\tau}\tau^{-0.45}, \quad A(r) = 0.093(1-r)/(1-r_0).$$

Figures 2–4 show the time dependences of the relative circumferential stress κ_2 at different fixed points of the FGM layer for kernels $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ (curves 1–4, respectively) with all other initial data (20) being identical. The figures also present the results for a spherical elastic FGM layer (curves 5) and for a homogeneous elastic spherical layer (curves 6) of the same dimensions. Figure 5 shows similar results for the relative radial stress κ_1. Everywhere negative stresses are compressive.

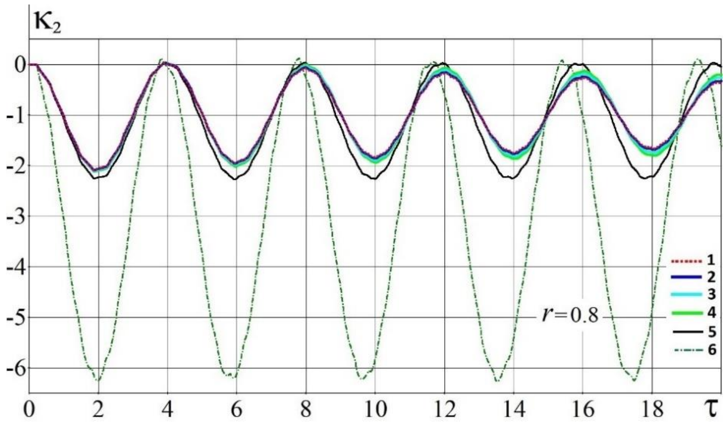

FIGURE 2. Evolution of κ_2 in time at the point $r = r_0 = 0.8$ (1 – with the regular kernel γ_0; 2 – 4 – with the singular kernels $\gamma_1, \gamma_2, \gamma_3$; 5 – for the elastic FGM; 6 – for the homogeneous elastic spherical layer)

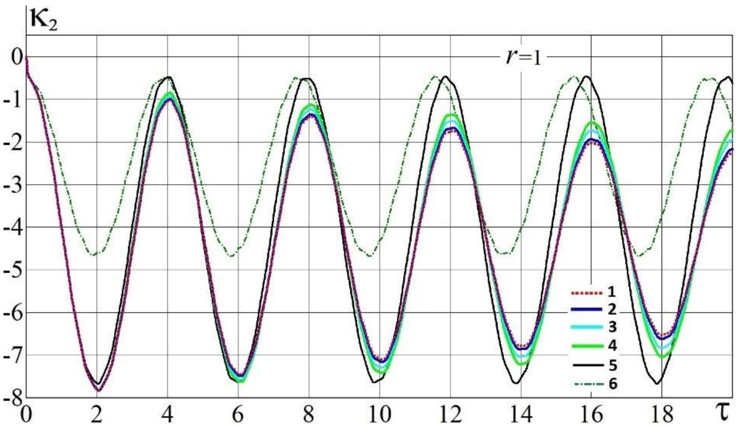

FIGURE 3. Evolution of κ_2 in time at the point $r = 1$

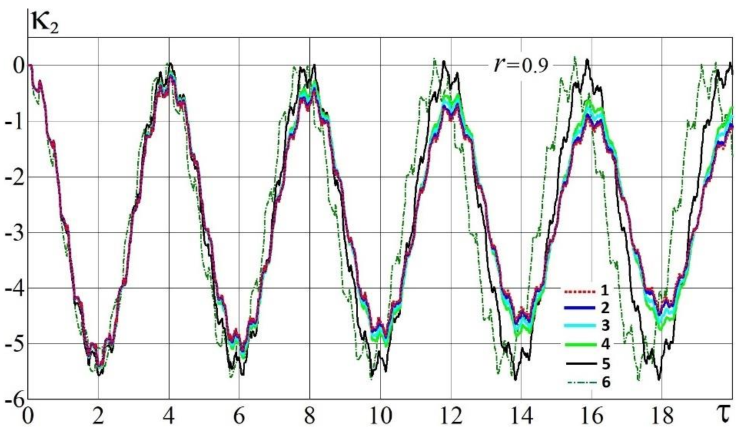

FIGURE 4. Evolution of κ_2 in time at the point $r = 0.9$

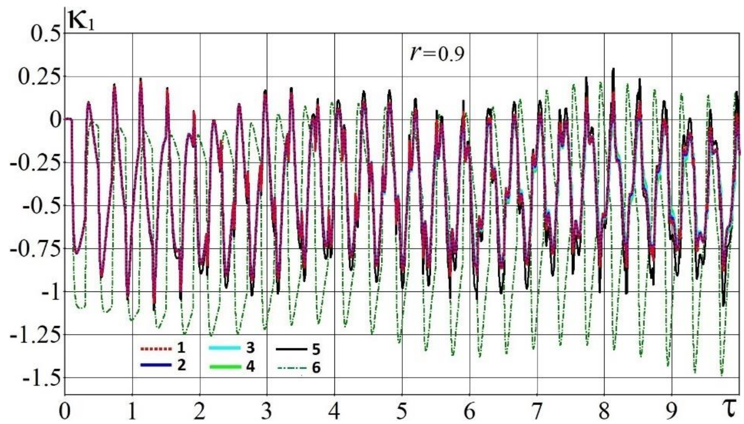

FIGURE 5. Evolution of κ_1 in time at the point $r = 0.9$

In the considered time range for viscoelastic FGMs, the similarity of nonstationary processes to each other is generally observed for all four shear relaxation kernels $\gamma_0, \gamma_1, \gamma_2, \gamma_3$. Curves 1 and 2 practically merge with each other. It should be noted that, as time progresses, the transient process for kernel γ_3 differs somewhat more from the processes for the other hereditary kernels. The figures also demonstrate the influence of viscosity and material inhomogeneity on the transient wave process.

CONCLUSION

Based on the integral Laplace transform in time and its reversal, new form of representation of solutions to non-stationary problems for inhomogeneous viscoelastic bodies is obtained within the framework of the Boltzmann-Volterra model for various types of hereditary kernels. This makes it possible in many cases to significantly simplify the dynamic calculations of various structures.

The influence of singular Rzhanitsyn-Koltunov relaxation kernels on transient processes in spherical viscoelastic FGM layer was investigated. It was found that hereditary kernels, related by the proposed special relations, in general have a similar effect on transient processes in viscoelastic materials.

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