Numerical solution of the bending equation for a three-layered plate with two parallel sides rigidly fixed and the other two parallel sides simply supported

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Abstract. The longitudinal vibration equation of a three-layer plate, with two parallel sides rigidly fixed and the other two parallel sides simply supported, is numerically solved using the finite difference method. First, the longitudinal vibration equations of the three-layer plate and the corresponding boundary conditions are expressed in finite-difference form. As a result, a system of algebraic equations is obtained. This system of algebraic equations is solved in Maple software for different step sizes. Based on the obtained solutions, the displacements and stresses in the layers of the three-layer elastic plate are determined. The calculated displacements and stresses are illustrated in graphs, and final conclusions are drawn.

Keywords. finite difference, layer, plate, external force, displacement

INTRODUCTION

Nowadays, three-layer plate-type structures are widely used in all fields of construction and engineering. Such plates are considered integral elements of various constructions and mechanisms. In these cases, determining the stresses that arise within them is one of the important problems of modern mechanics. Therefore, since the first half of the last century, many researchers have studied such problems and proposed various methods for their solutions. For example, the “Zig-Zag” theory [2], the generalized theory [3], the E. Reissner theory [4], and others.

However, to this day, the problem of determining the displacements and stresses in plates with rigidly clamped edges has not been completely solved. Studies devoted to the longitudinal vibrations of three-layer plates with two parallel edges clamped and the other two parallel edges free under dynamic stresses are relatively few. Research works [5–8] can be cited as examples of such studies. In all these studies, the vibration equations are solved using exact analytical methods. In works [9–12], the longitudinal and transverse vibration equations of three-layer plates have been derived. In the present work, the longitudinal and transverse vibration equations of a three-layer plate derived in [9–12] are numerically solved using the finite difference method.

FORMULATION OF THE PROBLEM

Let us consider a three-layer rectangular elastic plate located in the Cartesian coordinate system (Fig. 1), where one side has a length of l and the other side has a length of b. The parallel sides $x=0$ and $x=l$ of the rectangular three-layer elastic plate are rigidly clamped, while the other two parallel sides $y=0$ and $y=l$ are simply supported. The upper layer of the rectangular three-layer elastic plate has a thickness of h_1, the middle layer a thickness of $2h_0$, and the lower layer a thickness of h_2.

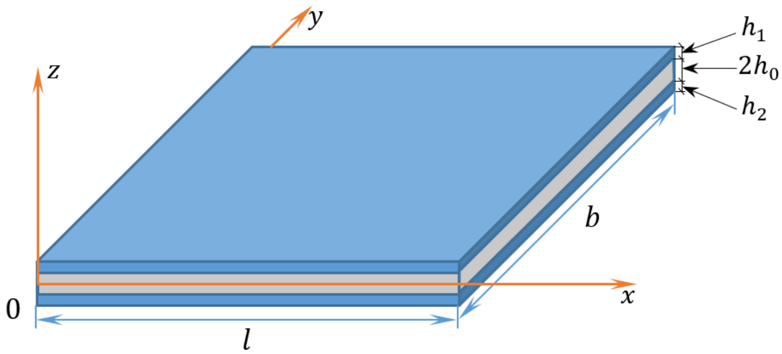


FIGURE 1. Rectangular three-layer elastic plate

In works [13–15], a system of longitudinal vibration equations for a rectangular three-layer elastic plate was derived. In the present study, we make use of this system of longitudinal vibration equations for the three-layer plate.

$$
\begin{aligned}
&N_1[\frac{\partial}{\partial x}W_0^{(0)}\left(x,t\right)]+K_1[U_0^{(0)}\left(x,t\right)]=F_1[f_x^{(1)}\left(x,t\right)],\\
&N_2[W_0^{(0)}\left(x,t\right)]+K_2[\frac{\partial}{\partial x}U_0^{(0)}\left(x,t\right)]=F_2[f_z^{(2)}\left(x,t\right)].
\end{aligned}
\tag{1}
$$

In this system of equations, the coefficients N_1, N_2, K_1, K_2, F_1 and F_2 are differential operators. Their forms are presented in detail in works [13–15]. The expressions of these differential operators include coefficients that depend on the geometric and physical properties of the rectangular three-layer elastic plate.

The side x=0 of the three-layer rectangular plate under investigation is rigidly clamped. Therefore, for this side, we have the following:

$$
W\left(0,t\right)=0,\ \frac{\partial}{\partial x}W\left(0,t\right)=0 \text{ va } U\left(0,t\right)=0,\ \frac{\partial}{\partial x}U\left(0,t\right)=0 \tag{2}
$$

The $x=a$ -side is also rigidly clamped:

$$
W\left(a,t\right)=0 \text{ va } \frac{\partial}{\partial x}W\left(a,t\right)=0\,;U\left(a,t\right)=0 \text{ va } \frac{\partial}{\partial x}U\left(a,t\right)=0 \tag{3}
$$

At time $t=0$ the three-layer rectangular elastic plate was in a state of rest. Therefore, the initial conditions are as follows:

$$
U\left(x,0\right)=\frac{\partial}{\partial x}U\left(x,0\right)=\frac{\partial^2}{\partial x^2}U\left(x,0\right)=0;\ \ W\left(x,0\right)=\frac{\partial}{\partial x}W\left(x,0\right)=\frac{\partial^2}{\partial x^2}W\left(x,0\right)=0. \tag{4}
$$

The longitudinal vibration problem of a three-layer elastic rectangular plate with two parallel edges hinged and fixed, and the other two parallel edges simply supported, was reduced to solving the system of differential equations (1) using the boundary conditions (2)–(4).

RESEARCH RESULTS

To solve the problem of longitudinal vibration of a three-layer elastic rectangular plate with two opposite sides rigidly hinged and the other two opposite sides freely supported, we express (1) the system of equations, as well as (2)–(3) boundary and (4) initial conditions in finite difference form. By expressing the system of equations (1) in finite difference form, we obtain the following:

$$
\begin{gathered}
n_{11}W_{i+1}^{j+2}+n_{12}W_{i-1}^{j+2}+n_{13}W_{i+1}^{j+1}+n_{14}W_{i-1}^{j+1}+n_{15}W_{i+1}^{j-1}+n_{16}W_{i-1}^{j-1}+n_{17}W_{i+1}^{j}+n_{18}W_{i-1}^{j}+n_{19}W_{i+1}^{j-2}+\\
+n_{110}W_{i-1}^{j-2}+n_{111}W_{i+2}^{j+1}+n_{112}W_{i+2}^{j}+n_{113}W_{i+2}^{j-1}+n_{114}W_{i-2}^{j+1}+n_{115}W_{i-2}^{j}+n_{116}W_{i-2}^{j-1}+n_{117}W_{i+3}^{j}+\\
+n_{118}W_{i-3}^{j}+k_{11}U_{i}^{j+2}+k_{12}U_{i}^{j+1}+k_{13}U_{i}^{j-1}+k_{14}U_{i}^{j}+k_{15}U_{i}^{j-2}+k_{16}U_{i+1}^{j+1}+k_{17}U_{i-1}^{j+1}+k_{18}U_{i+1}^{j}+\\
+k_{19}U_{i-1}^{j}+k_{110}U_{i+1}^{j-1}+k_{111}U_{i-1}^{j-1}+k_{112}U_{i+2}^{j}+k_{113}U_{i-2}^{j}=d_{11}\tilde{f}_{i}^{j+2}+d_{12}\tilde{f}_{i}^{j+1}+d_{13}\tilde{f}_{i}^{j-1}+d_{14}\tilde{f}_{i}^{j}+\\
+d_{15}\tilde{f}_{i}^{j-2}+d_{16}\tilde{f}_{i+1}^{j+1}+d_{17}\tilde{f}_{i-1}^{j+1}+d_{18}\tilde{f}_{i+1}^{j}+d_{19}\tilde{f}_{i-1}^{j}+d_{110}\tilde{f}_{i+1}^{j-1}+d_{111}\tilde{f}_{i-1}^{j-1}+d_{112}\tilde{f}_{i+2}^{j}+d_{113}\tilde{f}_{i-2}^{j};\\
n_{21}W_{i}^{j+2}+n_{22}W_{i}^{j+1}+n_{23}W_{i}^{j-1}+n_{24}W_{i}^{j}+n_{25}W_{i}^{j-2}+n_{26}W_{i+1}^{j+1}+n_{27}W_{i-1}^{j+1}+n_{28}W_{i+1}^{j}+\\
+n_{29}W_{i-1}^{j}+n_{210}W_{i+1}^{j-1}+n_{211}W_{i-1}^{j-1}+n_{212}W_{i+2}^{j}+n_{213}W_{i-2}^{j}+k_{21}U_{i+1}^{j+2}+k_{22}U_{i-1}^{j+2}+k_{23}U_{i+1}^{j+1}+\\
+k_{24}U_{i-1}^{j+1}+k_{25}U_{i+1}^{j-1}+k_{26}U_{i-1}^{j-1}+k_{27}U_{i+1}^{j}+k_{28}U_{i-1}^{j}+k_{29}U_{i+1}^{j-2}+k_{210}U_{i-1}^{j-2}+k_{211}U_{i+2}^{j+1}+k_{212}U_{i+2}^{j}+\\
+k_{213}U_{i+2}^{j-1}+k_{214}U_{i-2}^{j+1}+k_{215}U_{i-2}^{j}+k_{216}U_{i-2}^{j-1}=d_{21}\tilde{\tilde{f}}_{i}^{j+2}+d_{22}\tilde{\tilde{f}}_{i}^{j+1}+d_{23}\tilde{\tilde{f}}_{i}^{j-1}+d_{24}\tilde{\tilde{f}}_{i}^{j}+d_{25}\tilde{\tilde{f}}_{i}^{j-2}+\\
+d_{26}\tilde{\tilde{f}}_{i+1}^{j+1}+d_{27}\tilde{\tilde{f}}_{i-1}^{j+1}+d_{28}\tilde{\tilde{f}}_{i+1}^{j}+d_{29}\tilde{\tilde{f}}_{i-1}^{j}+d_{210}\tilde{\tilde{f}}_{i+1}^{j-1}+d_{211}\tilde{\tilde{f}}_{i-1}^{j-1}+d_{212}\tilde{\tilde{f}}_{i+2}^{j}+d_{213}\tilde{\tilde{f}}_{i-2}^{j};
\end{gathered}
\tag{5}
$$

We take the following initial conditions:

$$U_i^0=0;\ U_i^1=U_i^0;\ U_i^{-1}=-U_i^1;W_i^0=0;\ W_i^1=W_i^0;\ W_i^{-1}=-W_i^1. \tag{6}$$

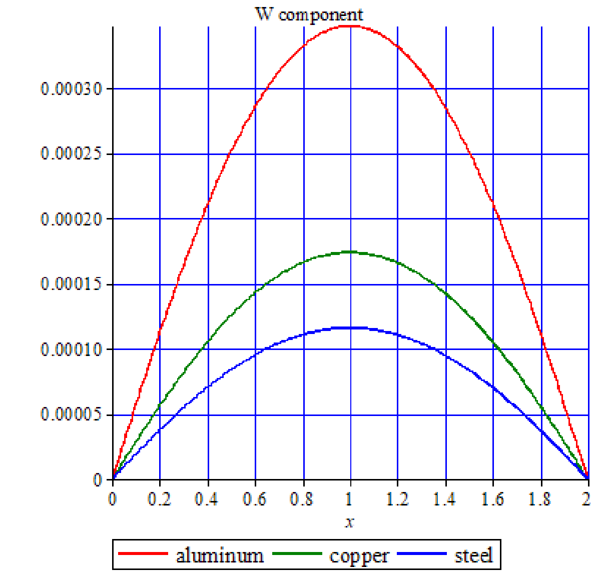
From the boundary conditions, we take the following:

$$U_0^j=0;\ U_1^j=U_0^j;\ W_0^j=0;\ W_1^j=W_0^j. \tag{7}$$

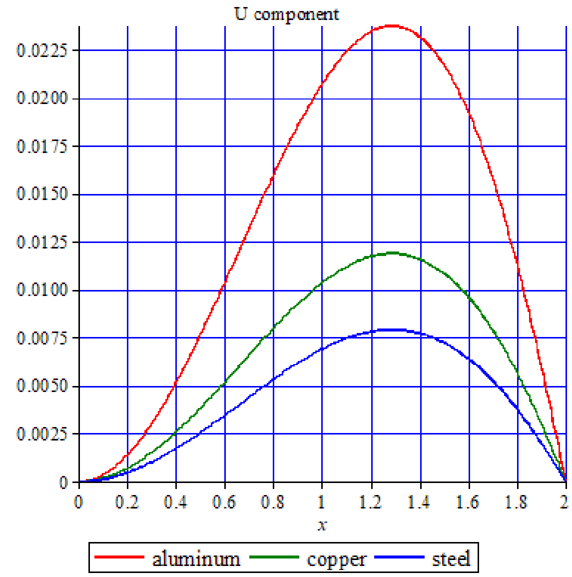
Thus, to solve the problem of longitudinal vibration of a three-layer elastic rectangular plate with two opposite sides rigidly hinged and the other two opposite sides freely supported, it is necessary to solve equation (5) taking into account conditions (6) and (7).

Practical issue. Now, to solve the practical problem, we specify the dimensions of the three-layer elastic rectangular plate with two opposite sides rigidly hinged and the other two opposite sides freely supported. The plate length is $1.2\,m$, and the thicknesses of the first and second outer layers are both $0.006\,m$. The middle layer thickness is $0.04\,m$. We consider the middle layer of the three-layer plate as polymer and the outer layers as aluminum. For the polymer material, the modulus of elasticity is $0.165\cdot10^9\,Pa$ and the Poisson's ratio is 0.03125. For aluminum, the modulus of elasticity is $69\cdot10^9\,Pa$ and the Poisson's ratio is 0.33. The density of the polymer material is $30\,kg/m^3$, and the density of aluminum is $2700\,kg/m^3$. A longitudinal force of $60\,N$ is applied to the surface of the three-layer plate with two opposite sides rigidly hinged and the other two opposite sides freely supported.

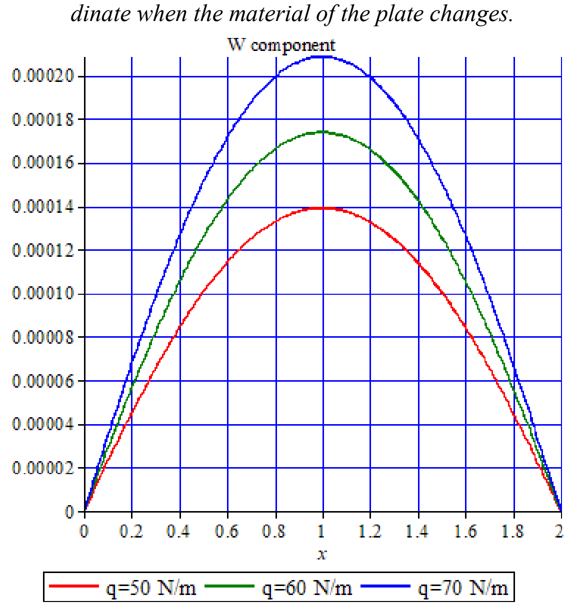
Using the given values in the practical problem and specifying the number of steps, we determine the coefficients of equation (5). We divide the time and coordinate intervals into 20 segments each. As a result, 400 equations are formed. These 400 equations contain 400 unknowns. We solve the resulting system of equations using the Maple software. From the obtained solutions, we plot the graphs of longitudinal and transverse displacements occurring at the points of the three-layer elastic rectangular plate with two opposite sides rigidly hinged and the other two opposite sides freely supported (see Figure 2).



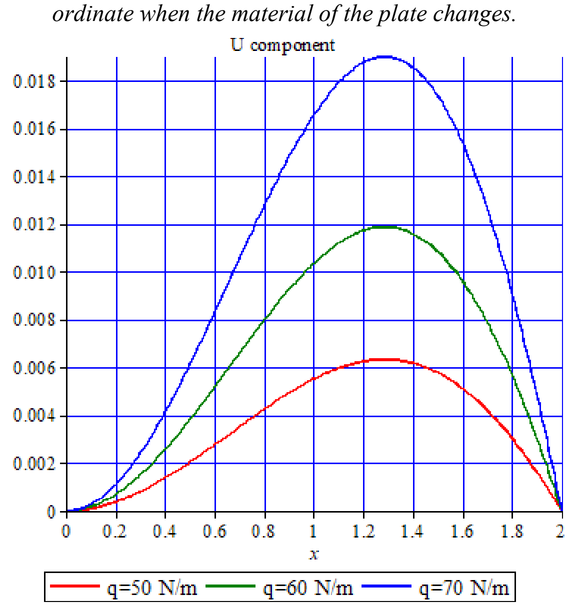
a) The graph of the displacement W as a function of the x-coordinate when the material of the plate changes.



b) The graph of the displacement U as a function of the x-coordinate when the material of the plate changes.



c) The graph of the displacement W as a function of the x-coordinate when the external force changes.



d) The graph of the displacement U as a function of the x-coordinate when the external force changes.

FIGURE 2. Transverse and longitudinal displacements of a three-layer plate.

In Figure 2, the displacement vector components U and W are shown. In Figure 2a, the graph of the W component of the displacement vector is presented as a function of the x-coordinate for cases where the materials of the outer layers of the plate are aluminum, copper, and steel, respectively. From the graph, it is evident that as the material of the outer layer of the three-layer plate becomes stiffer, the displacements decrease. In Figure 2b, the graph of the U displacement component versus the x-coordinate is shown for the same plate with outer layers made of aluminum, copper, and steel. The graph indicates that the longitudinal displacements are smaller at the beginning of the plate and increase towards the end. Figures 2b and 2c present graphs of the variation of the displacement vector components W and U along the x-coordinate for a three-layer plate with copper outer layers under distributed forces of 50, 60, and 70 N/m applied to its surface.

CONCLUSION

In Figure 2a, the value of the W component of the displacement vector for the three-layer plate with aluminum outer layers differs by 65.7% from that of the plate with steel outer layers. In Figure 2b, the value of the U component of the displacement vector for the plate with aluminum outer layers differs by 69% from that of the plate with steel outer layers, and by 52% from that of the plate with copper outer layers. This result is consistent with the physical nature of the problem. The reliability of the obtained solutions can also be observed from the graphs presented in Figures 2c and 2d. It can be seen from these figures that as the magnitude of the external load applied to the surface of the three-layer elastic rectangular plate—with two opposite sides rigidly hinged and the other two opposite sides freely supported—increases, the values of the W and U components of the displacement vector also increase. This fact further confirms the correctness of the obtained solution.

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