3rd International Conference Advanced Mechanics: Structure, Materials, Tribology

Application of the Mixed Approximation for Solving Two Problems of the Theory of Elasticity by the Finite Element Method

AIPCP25-CF-AMSMT2025-00038 | Article

Application of the Mixed Approximation for Solving Two Problems of the Theory of Elasticity by the Finite Element Method

Obid Abdullayev[1,a)], Zokir Khudoyberdiyev[2,b)], Khursand Ismoilov[3,c)], Bakhtiyor Usmonov[4,d)], Ruslan Butaev[5,e)].

[1,2,3]*Samarkand State University, Samarkand, Uzbekistan*
[4]*Samarkand State University Urgut Branch, Samarkand, Uzbekistan*
[5]*Jizzakh Branch of Mirzo Ulugbek National University of Uzbekistan Jizzakh, Uzbekistan*

[a)] obidabdullayev@gmail.com
[b)] Corresponding author: xudoyberdiyevz@mail.ru
[c)] kh.ismoilov89@gmail.com
[d)] usmonov@gmail.com
[e)] butayev@gmail.com

Annotation: The article formulates a discrete problem of the theory of elasticity using the mixed finite element method. Finite elements, a set of nodes, approximation domains, and a series of finite-dimensional subspaces to which the solutions of the discrete problem must belong are defined. An algorithm is provided for constructing piecewise polynomial functions for both displacements and stresses (strains). The existence and uniqueness of the solution to the discrete problem are proven.

Keywords: finite functions, triangulation, subspaces, mixed finite element method late

INTRODUCTION.

The mixed finite element method (MFEM) is based on the idea of simultaneously approximating displacements, stresses, and strains. This leads to an increase in the number of corresponding equations and a higher computational cost. Therefore, it becomes necessary to conduct studies aimed at exploring the capabilities of MFEM that is, the investigation of the existence, uniqueness, and stability of solutions, as well as assessing the method's effectiveness in solving specific problems in mechanics. The present work is devoted to the development of MFEM in the context of linear elasticity problems and its mathematical justification. Conditions for the solvability of the corresponding discrete problems are obtained [1-6].

STATEMENT OF THE PROBLEMS (ISSUE)

Let us consider a boundary value problem of linear elasticity in a generalized formulation [7].

$$(\varepsilon,\chi)_X=(Bu,\chi)_X,\quad \forall\chi\in X \tag{1}$$

$$(\sigma,\tau)_X=(D\varepsilon,\tau)_X-(D\zeta,\tau)_X,\quad \forall\tau\in X \tag{2}$$

$$(\sigma,B\text{v})_X=<\rho,\text{v}>_u.\quad \forall\text{v}\in U \tag{3}$$

To construct mixed finite element methods (MFEM), we perform an approximation, replacing the spaces U and X, where $u \in U$ and $\sigma, \varepsilon \in X$, with finite-dimensional subspaces $U_h \subset U$ and $X_h \subset X$. In this case, the parameter is $h \to 0$.

On the domain Ω, we perform the discretization Γ_h:

$$\Gamma_h = \left\{ \begin{array}{l} \bigcup\limits_{m=1,..,N_m(h)} \bar{\Omega}_m = \bar{\Omega};\ \Omega_m \cap \Omega_1 = \varnothing, \forall m \neq 1; \\ \max\limits_{m=1,...,N_m(h)} h_m \leq h;\ h_m \leq cs_m, \forall \Omega_m \end{array} \right\} \tag{4}$$

Here, the diameter of the circumscribed sphere around Ω_m is h_m, and the diameter of the inscribed sphere in Ω_m is denoted by s_m. The number of subsets $N_m(h)$ is denoted by c — a constant that does not depend on h.

Let the sets of nodes for displacements and stresses (strains) be denoted by ε_h and Y_h, respectively [8-9]:

$$\varepsilon_h = \left(X_\alpha\right)_{\alpha=1}^{N_\alpha(h)},\ Y_h = \left(X_\gamma\right)_{\gamma=1}^{N_\gamma(h)}. \tag{5}$$

The sets of piecewise polynomial functions for $x_\alpha \in \varepsilon_h$ and $x_\gamma \in Y_h$ are defined as follows: $\left(\varphi_{h\alpha}(\mathrm{x})\right)_{\alpha=1}^{N_\alpha(h)}$ and $\left(\psi_{h\alpha}(\mathrm{x})\right)_{\gamma=1}^{N_\gamma(h)}$, possessing certain properties.

$$\varphi_{h\alpha}\left(\mathrm{x}_\beta\right) = \delta_{\alpha\beta}, \quad \forall \mathrm{x}_\alpha, \mathrm{x}_\beta \in \varepsilon_h \tag{6}$$

$$\psi_{h\gamma}\left(\mathrm{x}_\mu\right) = \delta_{\gamma\mu}, \quad \forall \mathrm{x}_\gamma, \mathrm{x}_\mu \in Y_h \tag{7}$$

$$\operatorname{supp} \varphi_{h\alpha}(\mathrm{x}) = \left\{\mathrm{x} \middle| \mathrm{x} \in \Lambda_{h\alpha}\right\} \tag{8}$$

$$\operatorname{supp} \psi_{h\gamma}(\mathrm{x}) = \left\{\mathrm{x} \middle| \mathrm{x} \in \Lambda_{h\gamma}\right\} \tag{9}$$

the collection of subsets in Ω_m containing the nodes $\mathrm{x}_\alpha \in \varepsilon_h$ and $\mathrm{x}_\gamma \in Y_h$ is denoted by $\Lambda_{h\alpha}$ and $\Lambda_{h\gamma}$, respectively.

Using linear combinations of the above-mentioned basis functions, we define the functions for displacements and stresses (strains):

$$\mathrm{v}_{hi}(\mathrm{x}) = \sum_{\mathrm{x}_\alpha \in \varepsilon_h} \mathrm{v}_{hi}\left(\mathrm{x}_\alpha\right) \varphi_{h\alpha}(\mathrm{x}),\ \forall \mathrm{x} \in \Omega \tag{10}$$

$$\tau_{hij}(\mathrm{x}) = \sum_{\mathrm{x}_\gamma \in Y_h} \tau_{hij}\left(\mathrm{x}_\gamma\right) \psi_{h\gamma}(\mathrm{x}),\ \forall \mathrm{x} \in \Omega \tag{11}$$

They, in turn, are $\Phi_h(\Omega)$ and $\Psi_h(\Omega)$, which are finite element spaces satisfying the following conditions:

$$\Phi_h(\Omega) \subset H^1(\Omega),\ \psi_h(\Omega) \subset L_2(\Omega), \tag{12}$$

$$U_h = \left\{\mathrm{v}_h \middle| \mathrm{v}_h = \left(\mathrm{v}_{hi}\right)_{i=1}^n, \mathrm{v}_{hi} \in \Phi_h(\Omega), \mathrm{v}_{hi}\big|_{\Gamma_u} = 0\right\} \tag{13}$$

$$X_h = \left\{\tau_h \middle| \tau_h = \left(\tau_{hij}\right)_{i,j=1}^n, \tau_{hij} \in \psi_h(\Omega)\right\} \tag{14}$$

Therefore, (13) and (14) are the approximating spaces for displacements and stresses (strains), respectively, with the following conditions holding: $U_h \subset U$ and $X_h \subset X$.

Let us consider the finite-dimensional spaces V_h and Z_h. Their elements are the following basis functions: $\hat{\vartheta}_h = \left(\hat{\vartheta}_{\mathrm{h}\alpha}\right)_{\alpha=1}^{N_\alpha(h)}$ and $\hat{\tau}_h = \left(\hat{\tau}_{h\gamma}\right)_{\gamma=1}^{N_\gamma(h)}$, with $\hat{\vartheta}_{h\alpha} = \left(\hat{\vartheta}_{\mathrm{hi}\alpha}\right)_{i=1}^n$ and $\hat{\tau}_{h\gamma} = \left(\tau_{hij\gamma}\right)_{i,j=1}^n$.

It is assumed that $\hat{\vartheta}_h \in V_h$ and $\hat{\tau}_h \in Z_h$ hold if $P_h\hat{\vartheta}_h \in U_h$ and $R_h\hat{\tau}_h \in X_h$. Then, P_h and R_h are mapping operators from V_h to U_h and from Z_h to X_h, respectively. They are defined as follows:

$$\vartheta_{hi}(\mathrm{x}) = \left(\mathrm{P}_h\hat{\vartheta}_h\right)_i(\mathrm{x}) = \sum_{\mathrm{x}_\alpha \in \varepsilon_h} \vartheta_{hi\alpha}\varphi_{h\alpha}(\mathrm{x}),\ \forall \hat{\vartheta}_h \in V_l \tag{15}$$

$$\tau_{hij}(\mathrm{x}) = \left(R_h\hat{\tau}_h\right)_{ij}(\mathrm{x}) = \sum_{\mathrm{x}_\gamma \in Y_h} \tau_{hij\gamma}\psi_{h\gamma}(\mathrm{x}),\ \forall \hat{\tau}_h \in Z_h \tag{16}$$

From conditions (6) and (7), we obtain:

$$\vartheta_{hi}\left(x_{\alpha}\right)=\vartheta_{hi\alpha},\ \forall x_{\alpha}\in\varepsilon_{h} \tag{17}$$

$$\tau_{hij}\left(x_{\gamma}\right)=\tau_{hij\gamma},\ \forall x_{\gamma}\in Y_{h} \tag{18}$$

Thus, the following functions correspond to the numerical parameters (17) and (18):

$$\vartheta_{h}=P_{h}\widehat{\vartheta}_{h},\ \forall\widehat{\vartheta}_{h}\in V_{h} \tag{19}$$

$$\tau_{h}=R_{h}\widehat{\tau}_{h},\ \forall\widehat{\tau}_{h}\in Z_{h} \tag{20}$$

With the following formulas, we introduce a set of functions: $\left(\mu_{hi\alpha}\left(x\right)\right)_{\alpha=1}^{N_{\alpha}\left(h\right)}\left(i=1,...,n\right)$.

where

$$\mu_{hi\alpha}\left(x\right)=\frac{1}{2}\frac{\partial\phi_{h\alpha}\left(x\right)}{\partial x_{i}}=\begin{cases}\frac{1}{2}\frac{\partial\lambda_{h\alpha}^{(m)}\left(x\right)}{\partial x_{i}},\forall x\in\Lambda_{h\alpha}\\ 0,\forall x\,\overline{\in}\,\Lambda_{h\alpha}\end{cases} \tag{21}$$

The notation $\lambda_{h\alpha}^{(m)}\left(x\right)$ refers to polynomial coordinate functions within the bounds of Ω_{m}, satisfying the condition: $\lambda_{h\alpha}^{(m)}\left(x_{\beta}\right)=\delta_{\alpha\beta}$. Let Y_{h} denote the linear span of the functions $\left(\mu_{hi\alpha}\left(x\right)\right)_{\alpha=1}^{N_{\alpha}\left(h\right)}$:

$$\left.\begin{array}{c}Y_{h}=\left\{\overline{\tau}_{h}\middle|\overline{\tau}_{h}=\left(\overline{\tau}_{hij}\right)_{i,j=1}^{n},\overline{\tau}_{hij}=\overline{\tau}_{hij},\right.\\ \overline{\tau}_{hij}\left(x\right)=\sum\limits_{x_{\alpha}\in\varepsilon_{h}}v_{hi\alpha}\mu_{hj\alpha}\left(x\right)+v_{hj\alpha}\mu_{hi\alpha}\left(x\right),\forall\hat{v}\in V_{h}\end{array}\right\} \tag{22}$$

Based on the independence of $\left(\varphi_{h\alpha}\left(x\right)\right)_{\alpha=1}^{N_{\alpha}\left(h\right)}$, we obtain the independence of the functions $\left(\mu_{h\alpha}\left(x\right)\right)_{\alpha=1}^{N_{\alpha}\left(h\right)}$.

Therefore, the dependence of $Y_{h}=BU_{h}$ is interpreted by Y_{h} as a Hilbert space dependence. Then the linear operator $B_{h}=BP_{h}$ acts from U_{h} to Y_{h}, defined by the following relations:

$$\left(B_{h}\hat{v}_{h}\right)_{ij}=\sum_{x_{\alpha}\in\varepsilon_{h}}v_{hi\alpha}\mu_{hj\alpha}\left(x\right)+v_{hj\alpha}\mu_{hi\alpha}\left(x\right) \tag{23}$$

Y_{h} is the image of the operator B_{h}.

SOLUTION OF THE PROBLEM

Find $\left(u_{h},\sigma_{h},\varepsilon_{h}\right)\in U_{h}\times X_{h}\times X_{h}$ such that

$$\left(\varepsilon_{h},\chi_{h}\right)_{X}=\left(Bu_{h},\chi_{h}\right)_{X},\forall\chi_{h}\in\chi_{h}; \tag{24}$$

$$\left(\sigma_{h},\tau_{h}\right)_{X}=\left(\wp\varepsilon_{h},\tau_{h}\right)_{X},\forall\tau_{h}\in\chi_{h}; \tag{25}$$

$$\left(\sigma_{h},Bv_{h}\right)_{X}=<\rho,v_{h}>_{U},\forall v_{h}\in U_{h}; \tag{26}$$

An arbitrary function from U_{h} and X_{h} is uniquely represented in the form of (19) and (20). Then (24)–(26) can be represented in the following form:

$$\left(R_{h},R_{h},\hat{\chi}_{h}\right)_{X}=\left(B_{h}\hat{u}_{h},R_{h}\hat{\chi}_{h}\right)_{X}, \tag{27}$$

$$\left(R_{h},R_{h},\hat{\tau}_{h}\right)_{X}=\left(\wp R_{h}\hat{\varepsilon}_{h}-\wp\zeta,R_{h}\hat{\tau}_{h}\right)_{X}, \tag{28}$$

$$\left(R_{h},R_{h},\hat{v}_{h}\right)_{X}=<\rho,P_{h}\hat{v}_{h}>_{U}, \tag{29}$$

The adjoint operators P_{h}^{*}, R_{h}^{*} and B_{h}^{*} to the operators P_{h}, R_{h} and B_{h} are defined by the relations:

$$\widehat{\vartheta}_{h}\rightarrow<g,P_{h}\widehat{\vartheta}_{h}>_{U}=<P_{h}^{*}g,\widehat{\vartheta}_{h}>_{V_{h}};\forall g\in U^{*},\forall\widehat{\vartheta}_{h}\in V_{h}; \tag{30}$$

$$\hat{\chi}_{h}\rightarrow\left(\tau,R_{h}\hat{\chi}_{h}\right)_{X}=<R_{h}^{*}\tau,\hat{\chi}_{h}>_{Z_{h}};\forall\tau\in X^{*},\forall\hat{\chi}_{h}\in Z_{h}; \tag{31}$$

$$\widehat{\vartheta}_{h}\rightarrow\left(\tau,B_{h}\widehat{\vartheta}_{h}\right)_{X}=<B_{h}^{*}\tau,\widehat{\vartheta}_{h}>_{V_{h}};\forall\tau\in X^{*},\forall\widehat{\vartheta}_{h}\in V_{h}. \tag{32}$$

Next, we introduce the linear operator M_h:

$$\hat{\chi}_h \to \left(R_h\hat{\tau}, R_h\hat{\chi}_h\right)_X =< M_h\hat{\tau}_h, \hat{\chi}_h >_{Z_h};\ \forall\hat{\tau}_h, \hat{\chi}_h \in Z_h; \tag{33}$$

Then $M_h : Z_h \to Z_h^*$ is represented in the form:

$$M_h = R_h^* R_h \quad \text{в} \quad Z_h^* \tag{34}$$

In detail, we define the linear operator M_h as G_h:

$$\hat{\chi}_h \to \left(\wp R_h\hat{\tau}, R_h\hat{\chi}_h\right)_X =< G_h\tau, \hat{\tau}_h, \hat{\chi}_h >_{Z_h};\ \forall\hat{\tau}_h, \hat{\chi}_h \in Z_h; \tag{35}$$

where $G_h : Z_h \to Z_h^*$ is represented in the form:

$$G_h = R_h^* \wp R_h \quad \text{в} \quad Z_h^* \tag{36}$$

Next, we define the linear operator H_h:

$$\hat{\chi}_h \to \left(B_h\hat{v}_h, R_h\hat{\chi}_h\right)_X =< H_h\hat{v}_h, \hat{\chi}_h >_{Z_h};\ \forall\hat{v}_h \in V_h, \forall\hat{\chi}_h \in V_h; \tag{37}$$

where $H_h : V_h \to Z_h^*$ is represented in the form:

$$H_h = R_h^* B_h \quad \text{в} \quad Z_h^* \tag{38}$$

Let $H_h^{'}$ denote the transpose of the operator H_h, where $H_h^{'} : Z_h \to V_h^*$ is defined as follows:

$$\hat{v}_h \to \left(\hat{\tau}_h, H_h\hat{v}_h\right)_{Z_h} =< H'_h\,\hat{\tau}_h, \hat{\chi}_h >_{V_h};\ \forall\hat{\tau}_h \in Z_h, \forall\hat{v}_h \in V_h; \tag{39}$$

Then $H_h^{'}$ can be represented in the form:

$$H_h^{'} = B_h^{'} R_h \quad \text{в} \quad Z_h^* \tag{40}$$

Based on the definition of the operators M_h, H_h, G_h, $H_h^{'}$, equations (29)–(31) can be written in the form of operator equations:

$$M_h\hat{\varepsilon}_h = H_h\hat{u}_h \quad \text{в} \quad Z_h^* \tag{41}$$

$$M_h\hat{\sigma}_h = G_h\hat{\varepsilon}_h - \hat{\xi}_h \quad \text{в} \quad Z_h^* \tag{42}$$

$$H_h^{'}\hat{\sigma}_h = \hat{\rho}_h \quad \text{в} \quad V_h^* \tag{43}$$

In this case,

$$\hat{\xi}_h = R_h^* D\zeta \in Z_h^* \tag{44}$$

$$\hat{\rho}_h = P_h^* \rho \in V_h^* \tag{45}$$

Taking into account that G_h can be represented in the form:

$$G_h = M_h\hat{D}_h \quad \text{в} \quad Z_h^* \tag{46}$$

where $\hat{D}_h$ is the restriction of the operator D to Z_h, and the element $\hat{\xi}_h \in Z_h^*$ is defined by the formula:

$$\hat{\xi}_h = M_h\hat{D}_h\hat{\zeta}_h \in Z_h^* \tag{47}$$

The element $\hat{\zeta}_h \in Z_h$ is the restriction of $\zeta \in X$ to Y_h.

Taking into account the above formulas, the system of equations (41)–(43) can be written in the form:

$$M_h\hat{\varepsilon}_h = H_h\hat{u}_h \quad \text{в} \quad Z_h^* \tag{48}$$

$$\hat{\sigma}_h = \hat{D}_h\left(\hat{\varepsilon}_h - \hat{\xi}_h\right) \quad \text{в} \quad Z_h^* \tag{49}$$

$$H_h^{'}\hat{\sigma}_h = \hat{\rho}_h \quad \text{в} \quad V_h^* \tag{50}$$

The latter system can be represented as a single operator equation:

$$A_h\hat{u}_h = \hat{f}_h \quad \text{в} \quad V_h^* \tag{51}$$

In this case, the linear operator $A_h : V_h \to V_h^*$ is defined by the formula:

$$A_h = H_h^{'}\hat{D}_h M_h^{-1} H_h \quad \text{в} \quad V_h^* \tag{52}$$

The element $\hat{f}_h \in V_h^*$ is defined by the formula:ой:

$$\hat{f}_h = \hat{\rho}_h + H_h^{'}\hat{D}_h\hat{\zeta}_h \in V_h^* \quad \text{в} \quad V_h^* \tag{53}$$

Thus, equations (48)–(50) define the system of finite element equations for the problem of elasticity theory [2].

Let's consider the question of existence and uniqueness of the approximate solution. To this end, we define scalar products on the spaces Z_h and V_h.

$$\hat{f}_h = \hat{\rho}_h + H_h^{\cdot}\hat{D}_h\hat{\zeta}_h \in V_h^*\left(\hat{\tau}_h,\hat{\chi}_h\right)_{Z_h} = \left(R_h\hat{\tau}_h, R_h\hat{\chi}_h\right)_X;\ \forall \hat{\tau}_h,\hat{\chi}_h \in Z_h \ \text{в}\ V_h^* \tag{54}$$

$$\|\cdot\|_{Z_h} : \hat{\tau}_h \in Z_h \to \|\hat{\tau}_h\|_{Z_h} = \left(\hat{\tau}_h,\hat{\tau}_h\right)_{Z_h}^{1/2} \ \text{в}\ V_h^* \tag{55}$$

It follows that (55) is a norm on Z_h and that it is a Hilbert space isomorphic to X_h. According to (54) and the operator $M_h : Z_h \to Z_h^*$, we obtain:

$$\left(\hat{\tau}_h,\hat{\chi}_h\right)_{Z_h} = < M_h\hat{\tau}_h,\hat{\chi}_h >_{Z_h} = < \hat{\tau}_h, M_h\hat{\chi}_h >_{Z_h};\ \forall \hat{\tau}_h,\hat{\chi}_h \in Z_h \ \text{в}\ V_h^* \tag{56}$$

And this ensures that M_h is the operator of canonical isometry from Z_h into Z_h^*; then there exists M_h^{-1}, the inverse of M_h, called the Ross operator, and $M_h^{-1} : Z_h^* \to Z_h$ is unique. Defining in Z_h^* the scalar product:

$$\left(\hat{\xi}_h,\hat{\eta}_h\right)_{Z_h^*} = \left(M_h^{-1}\hat{\xi}_h, M_h^{-1}\hat{\eta}_h\right)_{Z_h^*} = < M_h^{-1}\hat{\xi}_h,\hat{\eta}_h >_{Z_h} = < \hat{\xi}_h, M_h^{-1}\hat{\eta}_h >_{Z_h};\ \forall \hat{\xi}_h,\hat{\eta}_h \in Z_h^* \tag{57}$$

Let us establish that Z_h^* is also a Hilbert space such that.

$$\|\cdot\|_{Z_h^*} : \hat{\xi}_h \in Z_h^* \to \|\hat{\xi}_h\|_{Z_h^*} = \left(\hat{\xi}_h,\hat{\xi}_h\right)_{Z_h^*}^{1/2} \tag{58}$$

Now we construct a scalar product on V_h; for this we establish a correspondence between X_h and Y_h. To each element $B\vartheta_h \in Y_h$, $\forall \vartheta_h \in U_h$ we associate its projection onto X_h. This association is an operator in X. Denote the operator that performs this by J_h. By definition J_h is the operator of orthogonal projection of Y_h onto X_h. The element $J_hB\vartheta_h$ is the orthogonal projection of $B\vartheta_h \in Y_h$, $\forall \vartheta_h \in U_h$ onto X_h. This means that the difference $J_hB\vartheta_h$ − $B\vartheta_h$ is orthogonal (with respect to the inner product in) X to any arbitrary element from X_h, i.e.

$$\left(J_hBv_h - Bv_h, \chi_h\right)_X = 0,\ \forall \chi_h \in X_h \tag{59}$$

Based on the definitions of the operators J_h and R_h, we have:

$$\left(R_h\hat{\tau}_h, R_h\hat{\chi}_h\right)_X = \left(B_h\hat{v}_h, R_h\hat{\chi}_h\right)_X,\ \forall \hat{\chi}_h \in Z_h \tag{61}$$

As is known, $M_h : Z_h \to Z_h^*$ and $H_h : V_h \to Z_h^*$, we have:

$$M_h\hat{\tau}_h = H_h\hat{v}_h \ \text{в}\ Z_h^* \tag{62}$$

Thus, $\hat{\tau}_h \in Z_h$ is uniquely determined by the formula:

$$\hat{\tau}_h = M_h^{-1}H_h\hat{v}_h \tag{63}$$

This makes it possible to define the operator J_h using the following expression:

$$\hat{\tau}_h = J_hBv_h = R_hM_h^{-1}H_h\hat{v}_h \tag{64}$$

Let us note some properties of the orthoprojector J_h. Based on (59), we have:

$$\left(J_hBv_h, Bw_h\right)_X = \left(J_hBv_h, J_hBw_h\right)_X = \left(Bv_h, J_hBw_h\right)_X,\ \forall v_h, w_h \in U_h \tag{65}$$

Next,

$$J_h^2Bv_h = J_h\left(J_hBv_h\right) = J_hBv_h \quad \text{в}\ X_h \tag{66}$$

Using (65) as well as the Cauchy–Bunyakovsky–Schwarz inequality, we find:

$$\|J_hBv_h\|_X^2 = \left(J_hBv_h, Bv_h\right)_X \le \|J_hBv_h\|_X \|Bv_h\|_X,\ \forall v_h \in U_h \tag{67}$$

it follows that

$$\|J_hBv_h\|_X \le \|Bv_h\|_X,\ \forall v_h \in U_h \tag{68}$$

the equality holds only on the common elements of X_h and Y_h. Based on (68), the norm of the operator J_h is:

$$\|J_h\|_{L(Y_h:X_h)} = \sup_{v_h \in U_h/\{0\}} \frac{\|J_hBv_h\|_X}{\|Bv_h\|_X} = 1 \tag{69}$$

Suppose

$$d_0 \le d(h) = \inf_{v_h \in U_h/\{0\}} \frac{\|J_hBv_h\|}{\|Bv_h\|} \tag{70}$$

Here, d_0 is a positive number independent of h. The given relations make it possible to define the scalar product on V_h:

$$\begin{aligned}(\hat{v}_h,\hat{w}_h)_{V_h} &= (J_h B_h \hat{v}_h, J_h B_h \hat{w}_h)_X = (J_h B_h \hat{v}_h, B_h \hat{w}_h)_X = (B_h \hat{v}_h, J_h B_h \hat{w}_h)_X = \\ &< M_h^{-1} H_h \hat{v}_h, H_h \hat{w}_h >_{Z_h} = < H_h \hat{v}_h, M_h^{-1} H_h \hat{w}_h >_{Z_h};\ \forall \hat{v}_h, \hat{w}_h \in V_h\end{aligned} \tag{71}$$

the norm in V_h is defined by the formula:

$$\|\cdot\|_{V_h} : \hat{v}_h \in V_h \to \|\hat{v}_h\|_{V_h} = (\hat{v}_h,\hat{v}_h)_{V_h}^{1/2} \tag{72}$$

The norm in V_h^* is defined by the formula:

$$\|\cdot\|_{V_h^*} : \hat{f}_h \in V_h^* \to \|\hat{f}_h\|_{V_h^*} = \sup_{\hat{v}_h \in V_h/\{0\}} \frac{|< \hat{f}_h, \hat{v}_h >_{V_h}|}{\|\hat{v}_h\|_{V_h}} \tag{73}$$

Based on the following relations:

$$\begin{aligned}(\hat{v}_h,\hat{w}_h)_{V_h} &=< J_h \hat{v}_h, \hat{w}_h >_{V_h} =< H_h^{'} M_h^{-1} H_h \hat{v}_h, \hat{w}_h >_{V_h} =< \hat{v}_h, H_h^{'} M_h^{-1} H_h \hat{w}_h >_{V_h} = \\ &=< \hat{v}_h, J_h \hat{w}_h >_{V_h};\ \forall \hat{v}_h, \hat{w}_h \in V_h\end{aligned} \tag{74}$$

It can be concluded that J_h is the operator of the canonical isometry from V_h to V_h^* and is represented as:

$$J_h = H_h^{'} M_h^{-1} H_h \tag{75}$$

Obviously, $J_h^{-1} : V_h^* \to V_h$ is the Riesz operator for V_h, and a scalar product can be introduced on V_h^*:

$$(\hat{f}_h, \hat{g}_h)_{V_h^*} = (J_h^{-1}\hat{f}_h, J_h^{-1}\hat{g}_h)_{V_h} =< J_h^{-1}\hat{f}_h, \hat{g}_h >_{V_h} =< \hat{f}_h, J_h^{-1}\hat{g}_h >_{V_h};\ \forall \hat{f}_h, \hat{g}_h \in V_h^* \tag{76}$$

and V_h^* is also a Hilbert operator/space. Based on the above relations, we will show that (51) has a unique solution. It is necessary to assume certain properties regarding the operator $\hat{D}_h$. We consider $\hat{D}_h$ to be a linear, self-adjoint, positive definite, and bounded operator from Z_h to Z_h^*, with the condition that:

$$\exists \delta_0 = const > 0 : \forall \hat{\tau}_h \in Z_h, (\hat{D}_h \hat{\tau}_h, \hat{\tau}_h)_{Z_h} \ge \delta_0 \|\hat{\tau}_h\|_{Z_h}^2 \tag{77}$$

$$\exists \Delta_0 = const < \infty : \forall \hat{\tau}_h \in Z_h, \|\hat{D}_h \hat{\tau}_h\|_{Z_h} \le \Delta_0 \|\hat{\tau}_h\|_{Z_h} \tag{78}$$

Taking into account that,

$$< A_h \hat{v}_h, \hat{w}_h >_{V_h} = (D_h M_h^{-1} H_h \hat{v}_h, M_h^{-1} H_h \hat{w}_h)_{V_h};\ \forall \hat{v}_h, \hat{w}_h \in V_h \tag{79}$$

and

$$\|\hat{v}_h\|_{V_h}^2 =< J_h \hat{v}_h, \hat{v}_h >_{V_h} = \|M_h^{-1} H_h \hat{v}_h\|_{Z_h}^2;\ \forall \hat{v}_h \in V_h \tag{80}$$

based on (79), we have:

$$< A_h \hat{v}_h, \hat{v}_h >_{V_h} \ge \delta_0 \|M_h^{-1} H_h \hat{v}_h\|_{Z_h}^2 = \delta_0 \|\hat{v}_h\|_{V_h}^2;\ \forall v_h \in V_h \tag{81}$$

Moreover, based on the Cauchy–Bunyakovsky–Schwarz inequality and (78), we obtain:

$$\|A_h \hat{v}_h\|_{V_h^*} \le \Delta_0 \|M_h^{-1} H_h \hat{v}_h\|_{Z_h} = \Delta_0 \|\hat{v}_h\|_{V_h};\ \forall v_h \in V_h \tag{82}$$

Thus, A_h is a positive definite and bounded operator V_h from V_h^* to.

Then there exists an operator A_h^{-1} acting from V_h^* to V_h such that

$$\|A^{-1}\|_{L(V_h^*:V_h)} \le \frac{1}{\delta_0} \tag{83}$$

This means that equation (51) has a unique solution that depends continuously on the right-hand side, i.e., on the element $\hat{f}_h \in V_h^*$, and we obtain the following estimates:

$$\|\hat{u}_h\|_{V_k} = \|A_h^{-1}\hat{f}_h\|_{V_k} \le \|A_h^{-1}\|_{D(V_k^*:V_k)} \|\hat{f}_h\|_{V_k^*} \le \frac{1}{\delta_0}\|\hat{f}_h\|_{V_k^*} \tag{84}$$

$$\|\hat{\varepsilon}_h\|_{Z_k} = \|M_h^{-1}H_h\hat{u}_h\|_{Z_h} \leq \|\hat{u}_h\|_{V_h} \leq \frac{1}{\delta_0}\|\hat{f}_h\|_{V_h^*} \tag{85}$$

$$\|\hat{\sigma}_h\|_{Z_h} = \left\|\hat{D}_h\left(\hat{\varepsilon}_h - \hat{\zeta}_h\right)\right\|_{Z_h} \leq \Delta_0\left(\|\hat{\varepsilon}_h\|_{Z_h} + \|\hat{\zeta}_h\|_{Z_h}\right) \leq \frac{\Delta_0}{\delta_0}\|\hat{f}_h\|_{V_h^*} + \Delta_0\|\hat{\zeta}_h\|_{Z_h} \tag{86}$$

As a result, we obtain the following estimates:

$$\|u_h\|_U \leq \frac{1}{\delta_0 d_0}\|\hat{f}_h\|_{V_h^*} \tag{87}$$

$$\|\varepsilon_h\|_x \leq \frac{1}{\delta_0}\|\hat{f}_h\|_{V_h^*} \tag{88}$$

$$\|\sigma_h\|_X \leq \frac{\Delta_0}{\delta_0}\|\hat{f}_h\|_{V_h^*} + \Delta_0\|\hat{\zeta}_h\|_{Z_h} \tag{89}$$

Since $U_h \subset U$ has this property, then by the embedding theorem and the equivalence of norms

$$\|u_h\|_{[L_2(\Omega)]^n} \leq c\|u_h\|_{[H^1(\Omega)]^n} \leq c_0\|u_h\|_U \leq \frac{c_0}{\delta_0 d_0}\|\hat{f}_h\|_{V_h^*} \tag{90}$$

Based on the formula for determining the element $\hat{f}_h \in V_h^*$ and the norm of $\|\cdot\|_{V_h^*}$, we obtain:

$$\|\hat{f}_h\|_{V_h^*} \leq \|\hat{\rho}_h\|_{V_h^*} + \Delta_0\|\hat{\zeta}_h\|_{Z_h} \tag{91}$$

As a result, estimates (84)–(86) take the following form:

$$\|\hat{u}_h\|_{V_h} \leq \frac{1}{\delta_0}\|\hat{\rho}_h\|_{V_h^*} + \frac{\angle_0}{\delta_0}\|\hat{\zeta}_h\|_{Z_h} \tag{92}$$

$$\|\hat{\varepsilon}_h\|_{Z_h} \leq \frac{1}{\delta_0}\|\hat{\rho}_h\|_{V_h^*} + \frac{\angle_0}{\delta_0}\|\hat{\zeta}_h\|_{Z_h} \tag{93}$$

$$\|\hat{\sigma}_h\|_{Z_h} \leq \frac{\Delta_0}{\delta_0}\|\hat{\rho}_h\|_{V_h^*} + \Delta_0\left(1 + \frac{\Delta_0}{\delta_0}\right)\|\hat{\zeta}_h\|_{Z_h} \tag{94}$$

Therefore, the solution of the system of equations (48)–(50) exists and depends continuously on $\left(\hat{\rho}_h, \hat{\zeta}_h\right) \in V_h^* \times Z_h$

.

It is not difficult to show that the problem (48)–(50) has a unique solution. Let $\left(\hat{u}_h^{(1)}, \hat{\sigma}_h^{(1)}, \hat{\varepsilon}_h^{(1)}\right) \in V_h \times Z_h \times Z_h$ and $\left(\hat{u}_h^{(2)}, \hat{\sigma}_h^{(2)}, \hat{\varepsilon}_h^{(2)}\right) \in V_h \times Z_h \times Z_h$ be two solutions. Substituting them into (48)–(50) and subtracting one set of equations from the other, we obtain:

$$\begin{cases} M_h\left(\tilde{\varepsilon}_h^{(1)} - \tilde{\varepsilon}_h^{(2)}\right) = H_h\left(\hat{u}_h^{(1)} - \hat{u}_h^{(2)}\right) \quad \text{в} \quad Z_h^* & (95) \\ \hat{\sigma}_h^{(1)} - \hat{\sigma}_h^{(2)} = \hat{D}_h\left(\tilde{\varepsilon}_h^{(1)} - \tilde{\varepsilon}_h^{(2)}\right) \quad \text{в} \quad Z_h & (96) \\ H_h\left(\hat{\sigma}_h^{(1)} - \hat{\sigma}_h^{(2)}\right) = 0 \quad \text{в} \quad V_h^* & (97) \end{cases}$$

Then, by virtue of (92)–(94), we have $\left(\hat{u}_h^{(1)}, \hat{\sigma}_h^{(1)}, \hat{\varepsilon}_h^{(1)}\right) \in \left(\hat{u}_h^{(2)}, \hat{\sigma}_h^{(2)}, \hat{\varepsilon}_h^{(2)}\right)$. Therefore, the problem (48)–(50) is well-posed.

CONCLUSION

In conclusion, we note that condition (70), used in the consideration of the existence, uniqueness, and stability of the solution to the discrete problem (48)–(50), also plays an important role in proving the convergence of the mixed approximation process. At the same time, the lower bound estimates of the quantity 11111 are constructed for each specific projection-grid scheme and often are not a trivial task. Attempting to ignore condition (70) leads to an ill-conditioned system of equations upon mesh refinement, from which the discrete solution is determined with unsatisfactory accuracy.

REFERENCES

1. A. Abdullaev, B. Usmonov, M.Hidoyatova, S. Togoynazarov, Some methods for solving a transport problem to determine the optimal logistics plan. Aip Conference ProceedingsOpen source preview, **3286**(1), 030005, (2025). DOI: 10.1063/5.0279998
2. A. A. Abdullaev, N. M. Safarbayeva, B. Z. Usmonov, On the unique solvability of a nonlocal boundary value problem with the poincaré condition. E3s Web of ConferencesOpen source preview, **401**, 03048 (2023). DOI: 10.1051/e3sconf/202340103048
3. B. I. Islomov, B. Z. Usmonov, Nonlocal Boundary-value Problem for Third Order Elliptic-hyperbolic Type Equation. Lobachevskii Journal of MathematicsOpen source preview, **41**(1), pp. 32–38, (2020). DOI: 10.1134/S1995080220010060
4. Kh. Khudoynazarov and Z. B. Khudoyberdiyev Unsteady vibrations of a three-layer plate with an asymmetric structure, *OP Conf. Ser.: Earth Environ. Sci.* **614** 012061 (2020). DOI:10.1088/1755-1315/614/1/012061
5. B. F. Yalgashev, E.A. Ismoilov, Z. B. Khudoyberdiyev. Torsional Vibrations of Layered Cylindrical Viscoelastic Shells and Rods // AIP Conference Proceedings 2637, 030023 (2022); https://doi.org/10.1063/5.0118588. Published Online: 20 October 2022
6. Z. B. Khudoyberdiyev, Sh. R. Yaxshiboyev, Bending of a Cantilever Beam Under the Influence of a Force Applied to its Tip. AIP Conference Proceedings, 060014, 3244(1), (2024). https://doi.org/10.1063/5.0241681
7. Z. Khudoyberdiyev, Sh. Khudayberdiyeva, Sh. Yakhshiboyev, A. Begjanov, AIP Conf. Proc. 3177, 050010 (2025) https://doi.org/10.1063/5.02944897
8. R. Khalmuradov, Kh. Ismoilov, Z. Khudoyberdiyev, B. Babajanov, AIP Conf. Proc. 3177, 050011 (2025) https://doi.org/10.1063/5.0295160
9. Z. Khudoyberdiyev, Z.a Suyunova, A. Begjanov, J. Khasanov, AIP Conf. Proc. 3177, 050012 (2025) https://doi.org/10.1063/5.029489

AIP Publishing

LICENSE TO PUBLISH AGREEMENT FOR CONFERENCE PROCEEDINGS

This License to Publish must be signed and returned to the Proceedings Editor before the manuscript can be published. If you have questions about how to submit the form, please contact the AIP Publishing Conference Proceedings office (confproc@aip.org). For questions regarding the copyright terms and conditions of this License, please contact AIP Publishing's Office of Rights and Permissions, 1305 Walt Whitman Road, Suite 300, Melville, NY 11747-4300 USA; Phone 516-576-2268; Email: rights@aip.org.

Article Title ("Work"):
Application of the Mixed Approximation for Solving Two
Problems of the Theory of Elasticity by the Finite Element Method

All Author(s):
Obid Abdullayev, Zokir Khudoyberdiyev, Khursand Ismoilov
Bakhtiyor Usmonov, Ruslan Butaev

Title of Conference: AMSMT2025

Name(s) of Editor(s) Valentin L. Popov

All Copyright Owner(s), if not Author(s):

(Please list all copyright owner(s) by name. In the case of a Work Made for Hire, the employer(s) or commissioning party(ies) are the copyright owner(s). For large groups of copyright owners, attach a separate list to this form.)

Copyright Ownership and Grant of Rights

For the purposes of this License, the "Work" consists of all content within the article itself and made available as part of the article, including but not limited to the abstract, tables, figures, graphs, images, and multimedia files, as well as any subsequent errata. "Supplementary Material" consists of material that is associated with the article but linked to or accessed separately (available electronically only), including but not limited to data sets and any additional files.

This Agreement is an Exclusive License to Publish not a Transfer of Copyright. Copyright to the Work remains with the Author(s) or, in the case of a Work Made for Hire, with the Author(s)' employer(s). AIP Publishing LLC shall own and have the right to register in its name the copyright to the proceedings issue or any other collective work in which the Work is included. Any rights granted under this License are contingent upon acceptance of the Work for publication by AIP Publishing. If for any reason and at its own discretion AIP Publishing decides not to publish the Work, this License is considered void.

Each Copyright Owner hereby grants to AIP Publishing LLC the following irrevocable rights for the full term of United States and foreign copyrights (including any extensions):

1. The exclusive right and license to publish, reproduce, distribute, transmit, display, store, translate, edit, adapt, and create derivative works from the Work (in whole or in part) throughout the world in all formats and media whether now known or later developed, and the nonexclusive right and license to do the same with the Supplementary Material.
2. The right for AIP Publishing to freely transfer and/or sublicense any or all of the exclusive rights listed in #1 above. Sublicensing includes the right to authorize requests for reuse of the Work by third parties.
3. The right for AIP Publishing to take whatever steps it considers necessary to protect and enforce, at its own expense, the exclusive rights granted herein against third parties.

Author Rights and Permitted Uses

Subject to the rights herein granted to AIP Publishing, each Copyright Owner retains ownership of copyright and all other proprietary rights such as patent rights in the Work.

Each Copyright Owner retains the following nonexclusive rights to use the Work, without obtaining permission from AIP Publishing, in keeping with professional publication ethics and provided clear credit is given to its first publication in an AIP Publishing proceeding. Any reuse must include a full credit line acknowledging AIP Publishing's publication and a link to the Version of Record (VOR) on AIP Publishing's site.

Each Copyright Owner may:

1. Reprint portions of the Work (excerpts, figures, tables) in future works created by the Author, in keeping with professional publication ethics.
2. Post the Accepted Manuscript (AM) to their personal web page or their employer's web page immediately after acceptance by AIP Publishing.
3. Deposit the AM in an institutional or funder-designated repository immediately after acceptance by AIP Publishing.
4. Use the AM for posting within scientific collaboration networks (SCNs). For a detailed description of our policy on posting to SCNs, please see our Web Posting Guidelines (https://publishing.aip.org/authors/web-posting-guidelines).
5. Reprint the Version of Record (VOR) in print collections written by the Author, or in the Author's thesis or dissertation. It is understood and agreed that the thesis or dissertation may be made available electronically on the university's site or in its repository and that copies may be offered for sale on demand.
6. Reproduce copies of the VOR for courses taught by the Author or offered at the institution where the Author is employed, provided no fee is charged for access to the Work.
7. Use the VOR for internal training and noncommercial business purposes by the Author's employer.
8. Use the VOR in oral presentations made by the Author, such as at conferences, meetings, seminars, etc., provided those receiving copies are informed that they may not further copy or distribute the Work.
9. Distribute the VOR to colleagues for noncommercial scholarly use, provided those receiving copies are informed that they may not further copy or distribute the Work.
10. Post the VOR to their personal web page or their employer's web page 12 months after publication by AIP Publishing.
11. Deposit the VOR in an institutional or funder-designated repository 12 months after publication by AIP Publishing.
12. Update a prior posting with the VOR on a noncommercial server such as arXiv, 12 months after publication by AIP Publishing.

Author Warranties

Each Author and Copyright Owner represents and warrants to AIP Publishing the following:

1. The Work is the original independent creation of each Author and does not infringe any copyright or violate any other right of any third party.
2. The Work has not been previously published and is not being considered for publication elsewhere in any form, except as a preprint on a noncommercial server such as arXiv, or in a thesis or dissertation.
3. Written permission has been obtained for any material used from other sources and copies of the permission grants have been supplied to AIP Publishing to be included in the manuscript file.
4. All third-party material for which permission has been obtained has been properly credited within the manuscript.
5. In the event that the Author is subject to university open access policies or other institutional restrictions that conflict with any of the rights or provisions of this License, such Author has obtained the necessary waiver from his or her university or institution.

This License must be signed by the Author(s) and, in the case of a Work Made for Hire, also by the Copyright Owners. One Author/Copyright Owner may sign on behalf of all the contributors/owners only if they all have authorized the signing, approved of the License, and agreed to be bound by it. The signing Author and, in the case of a Work Made for Hire, the signing Copyright Owner warrants that he/she/it has full authority to enter into this License and to make the grants this License contains.

1. The Author must please sign here (*except* if an Author is a U.S. Government employee, then please sign under #3 below):

Author(s) Signature	Print Name	Date
[signature]	Zokir Khudoyberdiyev	24.11.2025

2. The Copyright Owner (if different from the Author) must please sign here:

Name of Copyright Owner | Authorized Signature and Title | Date

3. If an Author is a U.S. Government employee, such Author must please sign below.
The signing Author certifies that the Work was written as part of his/her official duties and is therefore not eligible for copyright protection in the United States.

Name of U.S. Government Institution (e.g., Naval Research Laboratory, NIST)

Author Signature | Print Name | Date

PLEASE NOTE: NATIONAL LABORATORIES THAT ARE SPONSORED BY U.S. GOVERNMENT AGENCIES BUT ARE INDEPENDENTLY RUN ARE NOT CONSIDERED GOVERNMENT INSTITUTIONS. (For example, Argonne, Brookhaven, Lawrence Livermore, Sandia, and others.) Authors at these types of institutions should sign under #1 or #2 above.

If the Work was authored under a U.S. Government contract, and the U.S. Government wishes to retain for itself and others acting on its behalf, a paid-up, nonexclusive, irrevocable, worldwide license in the Work to reproduce, prepare derivative works from, distribute copies to the public, perform publicly, and display publicly, by or on behalf of the Government, please check the box below and add the relevant Contract numbers.

☐ Contract #(s) ______ [1.16.1]

LICENSE TERMS DEFINED

Accepted Manuscript (AM): The final version of an author's manuscript that has been accepted for publication and incorporates all the editorial changes made to the manuscript after submission and peer review. The AM does not yet reflect any of the publisher's enhancements to the work such as copyediting, pagination, and other standard formatting.

arXiv: An electronic archive and distribution server for research article preprints in the fields of physics, mathematics, computer science, quantitative biology, quantitative finance, and statistics, which is owned and operated by Cornell University, http://arxiv.org/.

Commercial and noncommercial scholarly use: *Noncommercial* scholarly uses are those that further the research process for authors and researchers on an individual basis for their own personal purposes. They are author-to-author interactions meant for the exchange of ideas. *Commercial* uses fall outside the author-to-author exchange and include but are not limited to the copying or distribution of an article, either in hard copy form or electronically, for resale or licensing to a third party; posting of the AM or VOR of an article by a site or service where an access fee is charged or which is supported by commercial paid advertising or sponsorship; use by a for-profit entity for any type of promotional purpose. Commercial uses require the permission of AIP Publishing.

Embargo period: The period of time during which free access to the full text of an article is delayed.

Employer's web page: A web page on an employer's site that highlights the accomplishments and research interests of the company's employees, which usually includes their publications. (See also: Personal web page and Scholarly Collaboration Network).

Exclusive License to Publish: An exclusive license to publish is a written agreement in which the copyright owner gives the publisher exclusivity over certain inherent rights associated with the copyright in the work. Those rights include the right to reproduce the work, to distribute copies of the work, to perform and display the work publicly, and to authorize others to do the same. The publisher does not hold the copyright to the work, which continues to reside with the author. The terms of the AIP Publishing License to Publish encourage authors to make full use of their work and help them to comply with requirements imposed by employers, institutions, and funders.

Full Credit Line: AIP Publishing's preferred format for a credit line is as follows (you will need to insert the specific citation information in place of the capital letters): "Reproduced from [FULL CITATION], with the permission of AIP Publishing." A FULL CITATION would appear as: Journal abbreviation, volume number, article ID number *or* page number (year). For example: Appl. Phys. Lett. 107, 021102 (2015).

Institutional repository: A university or research institution's digital collection of articles that have been authored by its staff and which are usually made publicly accessible. As authors are encouraged and sometimes required to include their published articles in their institution's repository, the majority of publishers allow for deposit of the Accepted Manuscript for this purpose. AIP Publishing also allows for the VOR to be deposited 12 months after publication of the Work.

Journal editorial office: The contact point for authors concerning matters related to the publication of their manuscripts. Contact information for the journal editorial offices may be found on the journal websites under the "About" tab.

Linking to the Version of Record (VOR): To create a link to your article in an AIP Publishing journal or proceedings, you need to know the CrossRef digital object identifier (doi). You can find the doi on the article's abstract page. For instructions on linking, please refer to our Web Posting Guidelines at https://publishing.aip.org/authors/web-posting-guidelines.

National Laboratories: National laboratories are sponsored and funded by the U.S. Government but have independent nonprofit affiliations and employ private sector resources. These institutions are classified as Federally Funded Research and Development Centers (FFRDCs). Authors working at FFRCs are not considered U.S. Government employees for the purposes of copyright. The Master Government List of FFRDCs may be found at http://www.nsf.gov/statistics/ffrdclist/.

Personal web page: A web page that is hosted by the author or the author's institution and is dedicated to the author's personal research interests and publication history. An author's profile page on a social media site or scholarly collaboration network site is *not* considered a personal web page. (See also: Scholarly Collaboration Network; Employer's web page).

Peer X-Press: A web-based manuscript submission system by which authors submit their manuscripts to AIP Publishing for publication, communicate with the editorial offices, and track the status of their submissions. The Peer X-Press system provides a fully electronic means of completing the License to Publish. A hard copy of the Agreement will be supplied by the editorial office if the author is unable to complete the electronic version of the form. (Conference Proceedings authors will continue to submit their manuscripts and forms directly to the Conference Editors.)

Preprint: A version of an author's manuscript intended for publication but that has not been peer reviewed and does not reflect any editorial input or publisher enhancements.

Professional Publication Ethics: AIP Publishing provides information on what it expects from authors in its "Statement of ethics and responsibilities of authors submitting to AIP Publishing journals" (http://publishing.aip.org/authors/ethics). AIP Publishing is also a member of the Committee on Publication Ethics (COPE) (http://publicationethics.org/), which provides numerous resources and guidelines for authors, editors, and publishers with regard to ethical standards and accepted practices in scientific publishing.

Scholarly Collaboration Network (SCN): Professional networking sites that facilitate collaboration among researchers as well as the sharing of data, results, and publications. SCNs include sites such as Academia.edu, ResearchGate, and Mendeley, among others.

Supplementary Material: Related material that has been judged by peer review as being relevant to the understanding of the article but that may be too lengthy or of too limited interest for inclusion in the article itself. Supplementary Material may include data tables or sets, appendixes, movie or audio clips, or other multimedia files.

U.S. Government employees: Authors working at Government organizations who author works as part of their official duties and who are not able to license rights to the Work, since no copyright exists. Government works are in the public domain within the United States.

Version of Record (VOR): The final published version of the article as it appears in the printed journal/proceedings or on the Scitation website. It incorporates all editorial input, is formatted in the publisher's standard style, and is usually viewed in PDF form.

Waiver: A request made to a university or institution to exempt an article from its open-access policy requirements. For example, a conflict will exist with any policy that requires the author to grant a nonexclusive license to the university or institution that enables it to license the Work to others. In all such cases, the Author must obtain a waiver, which shall be included in the manuscript file.

Work: The "Work" is considered all the material that comprises the article, including but not limited to the abstract, tables, figures, images, multimedia files that are directly embedded within the text, and the text itself. The Work does not include the Supplementary Material (see Supplementary Material above).

Work Made for Hire: Under copyright law, a work prepared by an employee within the scope of employment, or a work that has been specially ordered or commissioned for which the parties have agreed in writing to consider as a Work Made for Hire. The hiring party or employer is considered the author and owner of the copyright, not the person who creates the work.