Stress State During Torsional Vibrations of a Truncated Circular Conical Shell in an Elastic Medium

Khursand Ismoilov[1, a)], Zokir Khudoyberdiyev[2, b)], Amirbek Begjanov[3, c)]

[1,2]Department of theoretical and engineering mechanics Samarkand State University, 140104, University blv. 15, Samarkand city, Uzbekistan.
[3]Department of Algebra and mathematical engineering Urgench State University, Urgench, Uzbekistan.

[a)] Corresponding author: kh.ismoilov89@gmail.com
[b)] xudoyberdiyevz@mail.ru
[c)] amirbek_beg@mail.ru

Abstract. Truncated conical shells located in the environment are used in many areas of industry and construction. Therefore, determining their strength, determining the displacements and stresses of the cross-sectional points are considered urgent problems of mechanics. In this work, the refined equation derived by the authors is adopted as the torsional vibration equation of a circular truncated conical shell and the equation is solved using the finite difference method. The shear stresses on the conical surfaces are determined using the finite difference method and depicted in graphs.

INTRODUCTION

Circular truncated conical shells are one of the main elements of engineering structures. Determining the displacements and stresses in them is one of the current issues of modern mechanics. Determining the vibration of truncated conical structures, or FSC, is a complex issue. Mathematical modeling of torsional vibrations of elastic conical shells, which takes into account the forces on the inner and outer surfaces of the conical shells, is a more complex problem [1-4].

The study of the dynamic state of conical shells by deriving vibration equations is carried out based on the physical and mechanical properties of the material [5-7]. The development of a mathematical model of the vibrational motion process and the creation of a calculation algorithm provide a basis for drawing more precise conclusions about the process [8-11]. Most research studies are conducted based on refined equations of elasticity theory rather than classical theory [12].

Therefore, in [13], the equations of torsional vibration of a circularly truncated conical shell were derived based on the refined equations of elastic theory. In this work [14-15], the torsional vibration equations of a circularly truncated conical shell are numerically solved. A conical shell with an inner radius r_0 and a thickness h at a section z=0. Inner and outer radii of a conical shell $r_1 = r_0 + fz$ and $r_2 = r_0 + h + fz$.

PROBLEM FORMULATION

We consider a circular truncated conical shell in a cylindrical coordinate system $Or\theta z$. We consider the circular truncated conical shell to be located in a deformable medium. The angle between the axis of symmetry and the truncated conical shell is α (Figure 1). In this case, the components of the displacement and deformation tensor will not depend on the particle θ. In this case, U_θ of the displacement tensor components, $\varepsilon_{r\theta}$, $\varepsilon_{z\theta}$ of the strain tensor components, and $\sigma_{r\theta}, \sigma_{z\theta}$ of the stress tensor components are nonzero. To derive the torsional vibration equations of a circular truncated conical shell, we represent the displacements, deformations, and stresses in terms of ψ potential functions. Substituting the expressions of the stresses in terms of potential functions into the equations of motion, we obtain the following.

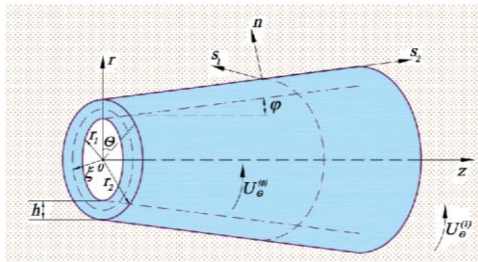

FIGURE 1. Conical shell in an elastic medium

$$\mu(\Delta\Psi_m) = \rho\ddot{\Psi}_m, \quad m = 0,1. \tag{1}$$

Boundary conditions

$$\sigma_{r\theta}^{(0)} - f\sigma_{z\theta}^{(0)} = (1+f^2) f_{ns_1}^{(i)}(z,t), \qquad r = r_1$$

$$\sigma_{r\theta}^{(0)} - f\sigma_{z\theta}^{(0)} = \sigma_{r\theta}^{(1)} - f\sigma_{z\theta}^{(1)} + (1+f^2) f_{ns_1}^{(i)}(z,t), \qquad r = r_2 \tag{2}$$

The initial condition is as follows:

$$U_\theta^{(0)}(r,z,t)\Big|_{r=r_2} = U_\theta^{(1)}(r,z,t)\Big|_{r=r_2} \tag{3}$$

The problem thus posed was reduced to solving eq. (1) with (2) boundary conditions and (3) initial conditions.

SOLUTION OF THE PROBLEM

By solving the problem, a system of differential equations for torsional vibrations of a circularly truncated conical shell located in a deformable medium is derived [15]. The complete derivation process is presented in the work.

$$\begin{gathered}
\left[a_{11}\frac{\partial^2}{\partial t^2} + a_{12}\frac{\partial^2}{\partial z^2} + a_{13}\frac{\partial}{\partial z}\right]U_{\theta 0}^{(0)} + \left\{b_{11}\frac{\partial^3}{\partial z\partial t^2} + b_{12}\frac{\partial^3}{\partial z^3} + \right.\\
\left. + b_{13}\frac{\partial^2}{\partial t^2} + b_{14}\frac{\partial^2}{\partial z^2} + b_{15}\frac{\partial}{\partial z} + b_{16}\right\}U_{\theta 0}^{(1)} = \left[1+f^2\right]\mu_0^{-1}\left[f_{ns_1}^{(i)}(z,t)\right];\\
\left[a_{21}\frac{\partial^2}{\partial t^2} + a_{22}\frac{\partial^2}{\partial z^2} + a_{23}\frac{\partial}{\partial t} + a_{24}\frac{\partial}{\partial z} + a_{25}\right]U_{\theta 0}^{(0)} + \left\{b_{21}\frac{\partial^3}{\partial z\partial t^2} + b_{22}\frac{\partial^3}{\partial t\partial z^2} + b_{23}\frac{\partial^3}{\partial z^3} + \right.\\
\left. + b_{24}\frac{\partial^2}{\partial t^2} + b_{25}\frac{\partial^2}{\partial z^2} + b_{26}\frac{\partial}{\partial t} + b_{27}\frac{\partial}{\partial z} + b_{28}\right\}U_{\theta 0}^{(1)} = \left[1+f^2\right]\mu_0^{-1}\left[f_{ns_2}^{(i)}(z,t)\right];
\end{gathered} \tag{4}$$

Here, coefficients a_{ij}, b_{ij} are constants depending on the geometric dimensions and physical parameters of the material. For example,

$$a_{11} = \frac{r_1^2}{4};\ldots.\ b_{11} = \frac{r_1^2}{8}\left(\ln r_1 - \frac{1}{4}\right);\ldots$$

This equation (4) represents the equation of torsional vibration of a circular truncated conical shell located in a deformable medium. Using the finite difference method, we arrive at the following system of algebraic equations.

$$\begin{gathered}
A_{11}U_{i,j+1}^{(0)} + A_{12}U_{i,j}^{(0)} + A_{13}U_{i,j-1}^{(0)} + A_{14}U_{i+1,j}^{(0)} + A_{15}U_{i-1,j}^{(0)} + B_{11}U_{i,j+1}^{(1)} + B_{12}U_{i,j}^{(1)} +\\
+B_{13}U_{i,j-1}^{(1)} + B_{14}U_{i+1,j+1}^{(1)} + B_{15}U_{i+1,j}^{(1)} + B_{16}U_{i+1,j-1}^{(1)} + B_{17}U_{i-1,j+1}^{(1)} +\\
+B_{18}U_{i-1,j}^{(1)} + B_{19}U_{i-1,j-1}^{(1)} + B_{110}U_{i+2,j}^{(1)} = \left(1+f^2\right)\mu_0^{-1}f_{ns_1}^{(1)}(z,t);\\
A_{21}U_{i,j+1}^{(0)} + A_{22}U_{i,j}^{(0)} + A_{23}U_{i,j-1}^{(0)} + A_{24}U_{i+1,j}^{(0)} + A_{25}U_{i-1,j}^{(0)} + B_{21}U_{i,j+1}^{(1)} +\\
+B_{22}U_{i,j}^{(1)} + B_{23}U_{i,j-1}^{(1)} + B_{24}U_{i+1,j+1}^{(1)} + B_{25}U_{i+1.j}^{(1)} + B_{26}U_{i+1.j-1}^{(1)} +\\
+B_{27}U_{i-1,j+1}^{(1)} + B_{28}U_{i-1,j}^{(1)} + B_{29}U_{i-1,j-1}^{(1)} + B_{210}U_{i+2,j}^{(1)} = \left(1+f^2\right)\mu_0^{-1}f_{ns_1}^{(2)}(z,t).
\end{gathered} \tag{5}$$

Where A_{ij} and B_{ij} are coefficients and time and coordinate steps and are constants depending on.

For example,

$$A_{11}=\frac{a_{11}}{\tau^2};\ A_{12}=-\frac{2a_{11}}{\tau^2}-\frac{2a_{12}}{h^2}-\frac{a_{13}}{h};.....$$

Similarly, we express the boundary conditions in finite difference terms as;

$$\begin{aligned}&U_{i,0}^{(0)}=0;\ U_{i,1}^{(0)}-U_{i,0}^{(0)}=0;\ U_{i,1}^{(0)}+2U_{i,0}^{(0)}-U_{i,-1}^{(0)}=0;\\&U_{i,0}^{(1)}=0;\ \ U_{i,1}^{(1)}-U_{i,0}^{(1)}=0;\ \ U_{i,1}^{(1)}-U_{i,0}^{(1)}=0.\end{aligned}\tag{6}$$

Boundary conditions at $z=0$

$$U_{1,j}^{(0)}-U_{0,j}^{(0)}=-\frac{h\cdot M_0}{\mu_m I};\quad U_{1,j}^{(1)}-U_{0,j}^{(1)}=-\frac{h\cdot M_0}{\mu_m I};\tag{7}$$

Boundary conditions at $z=l$

$$U_{N,j}^{(0)}=0;\ U_{N,j}^{(1)}=0;\quad U_{1,j}^{(0)}-U_{0,j}^{(0)}=0;\quad U_{1,j}^{(1)}-U_{0,j}^{(1)}=0.\tag{8}$$

We solve the system of eqs. (5) – (8) together. We divide the time and coordinate interval into 20 step segments. To determine the displacements and stresses at the points of the cross section of the circular truncated conical shell, we also express the displacements and stresses in finite difference form:

$$U_\theta=rU_{\theta 0}^{(0)}+\frac{U_{\theta 0}^{(1)}}{r}+\frac{r}{2}\ln r\frac{\partial^2U_{\theta 0}^{(1)}}{\partial t^2}-\frac{r}{2}\ln r\frac{\partial^2U_{\theta 0}^{(1)}}{\partial z^2}\ .\tag{9}$$

$$\begin{aligned}\sigma_{r\theta}^{(0)}&=\frac{\mu_0r^2}{4}\frac{\partial^2U_{\theta 0}^{(0)}}{\partial t^2}-\frac{\mu_0r^2}{4}\frac{\partial^2U_{\theta 0}^{(0)}}{\partial z^2}-\frac{\mu_0r^2}{4}\left(\ln r-\frac{1}{4}\right)\frac{\partial^4U_{\theta 0}^{(1)}}{\partial t^2\partial z^2}+\\&+\frac{\mu_0r^2}{8}\left(\ln r-\frac{1}{4}\right)\frac{\partial^4U_{\theta 0}^{(1)}}{\partial z^4}+\frac{\mu_0}{2}\frac{\partial^2U_{\theta 0}^{(1)}}{\partial t^2}-\frac{\mu_0}{2}\frac{\partial^2U_{\theta 0}^{(1)}}{\partial z^2}-\frac{2\mu_0}{r^2}U_{\theta 0}^{(1)};\\\sigma_{z\theta}^{(0)}&=\mu_0r\frac{\partial U_{\theta 0}^{(0)}}{\partial z}+\frac{\mu_0}{r}\frac{\partial U_{\theta 0}^{(1)}}{\partial z}+\frac{\mu_0r}{2}\ln r\frac{\partial^3U_{\theta 0}^{(1)}}{\partial z\partial t^2}-\frac{\mu_0r}{2}\ln r\frac{\partial^3U_{\theta 0}^{(1)}}{\partial z^3}\end{aligned}\tag{10}$$

RESULTS AND DISCUSSION

To solve the system of eqs. (5) –(8) together, we use the mathematical software package Maple 17. We assume the geometric and physical-mechanical characteristics of the conical shell and the external environment as follows;

The shell length is 0.8*m*, yhe inner radius 0.03*m,* the thickness is 0.005*m,* the angle of deflection is 2^0. The material of the conical shell is taken as steel ($E=2\cdot10^{11}$ Pa; $v=0.25$; $\rho=7850$ kg/m^3), in the first case, and aluminum ($E=0.7\cdot10^{11}$ Pa; $v=0.35$; $\rho=2750$ kg/m^3) in the second case. We assume that the deformable external environment is sand. For sand ($E=5\cdot10^7$ Pa; $v=0.3$; $\rho=2000$ kg/m^3)

Figure 2. shows a graph of the z-coordinate variation of the displacement vector component for different values of the torque applied to the $z=0$ end of a circularly truncated conical steel shell located in a sand environment when the angle This graph Fig. 2. shows that when the value of the torque is $5\,kNm$, the maximum value of the displacement of U_θ is 0.002, when the value of the torque is $15\,kNm$, the maximum value of the displacement of U_θ is 0.0062, and when the value of the torque is $25\,kNm$, the maximum value of the displacement U_θ is 0.013. of attack is 2^0.

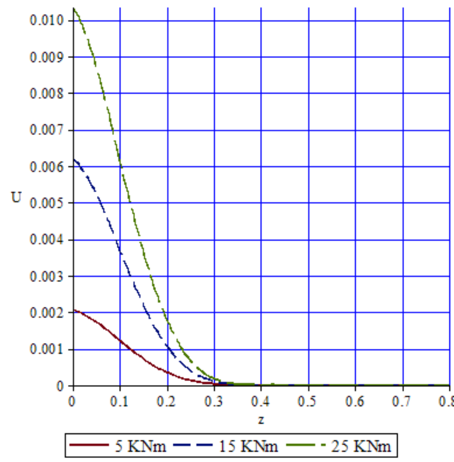

FIGURE 2. Graph of the vector of the displacement component U_θ of a steel shell in sand as a function of the *z* coordinate at different values of the torque

At all values of the torque, the graphs of the U_θ displacement vector components begin to fade after passing through the section z=0.3. Now, to make the results more reliable, we will assume that the shell material is aluminum (Fig.3). Here too, we apply torques of $5\,kNm$, $15\,kNm$, $25\,kNm$ and 1, respectively, to the free end of the conical shell at z=0.

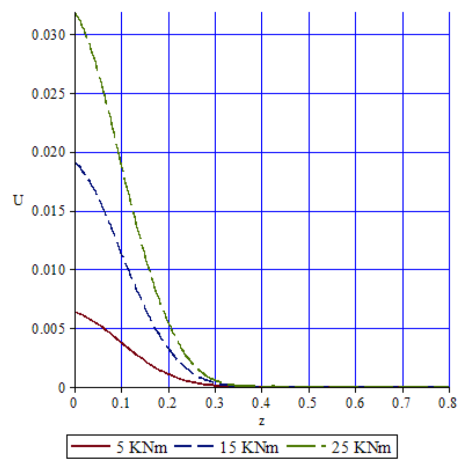

FIGURE 3. Graph of the variation of the displacement component of an aluminium shell U_θ in sand as a function of the vector *z* coordinate at different values of the torque

The modulus of elasticity of the aluminum material shell is softer than that of the steel material. Therefore, the value of component U_θ of the displacement vector should be larger when the shell material is aluminum. Fig. 3. shows that the values of the displacement vector component are 0.0053 at the value of the torque $5\,kNm$, 0.018 at the value of $15\,kNm$, and 0.033 at the value of $25\,kNm$.

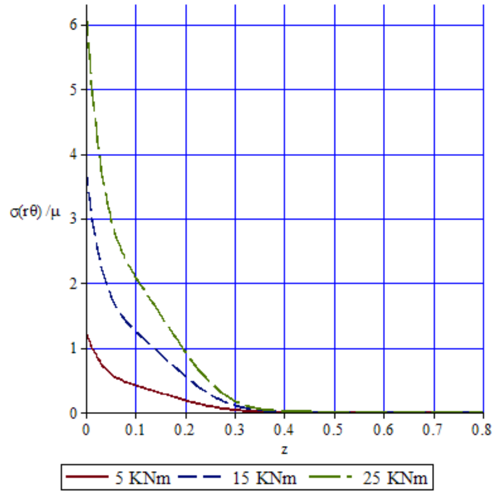

FIGURE 4. Graph of the variation of stresses in a shell aluminium in a sand environment as a function of the *z* coordinate.

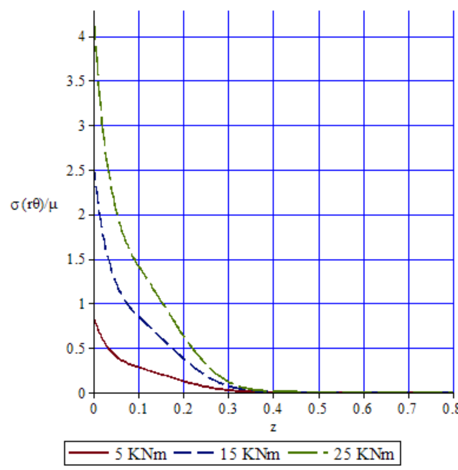

FIGURE 5. Graph of the variation of stresses in a shell steel in a sand environment as a function of the *z* coordinate.

It can be seen from these that the values of displacement U_θ obtained when the shell material is aluminum are significantly larger than those obtained when the shell material is steel. For example, it can be seen that the value of the displacement of a conical shell made of steel at a torque of $15kNm$ is almost three times smaller than the value of the displacement of an aluminum shell.

Figure 4. shows the graph of the change of the stress vector component $\sigma_{r\theta}^{(0)}$ as a function of the *z* coordinate when the material of the circular truncated conical shell is aluminum and the external environment is sand. In this case, the angle of attack was 2^0 and the values of the turning moment increased. As the value of the turning moment increased, the points of the circular truncated conical shell had more stress.

Figigure 5. shows the graph of the change of the stress vector component $\sigma_{r\theta}^{(0)}$ as a function of the *z* coordinate when the material of the circular truncated conical shell was copper. Figure 5. also shows that as the turning moment applied to the free end of the circular truncated conical shell *z*=0 increased, the stresses also increased.

CONCLUSIONS

The torsional vibration equation of a circular truncated conical shell located in a deformable medium was solved using the finite difference method. The displacements and stresses at the points of the truncated conical shell were depicted in graphs. From the graphs, it was concluded that the greater the density of the circular truncated conical shell material, the smaller the displacements and stresses in it.

REFERENCES

1. Kh. Khudoynazarov, Longitudinal-radial vibrations of a viscoelastic cylindrical three-layer structure. Facta Universitatis, Series, Mechanical Engineering, **22**(3), (2024). https://doi.org/10.22190/FUME231219010K
2. Y. Fang, Ch. Fan, Sh.F. Huang and Ch. Lui, Experimental study of active earth pressures on retaining walls adjacent to inclined rock faces. Journal of Geoengineering, **17**(4), 207-220, (2022).
3. Kh. Khudoynazarov, Tomsk State University Journal of Mathematics and Mechanics. **84**. pp. 152–166. (2023). doi: 10.17223/19988621/84/12
4. Kh. Khudoynazarov , *Journal of Samara State Technical University*, ser. Physical and Mathematical Sciences, Vol. **27**, No. 4, 2023, pp 702–722. (2002). https://doi.org/10.14498/vsgtu
5. S. Zhiyong, C.Qingjie, D.Qiyi, International Journal of Mechanical Sciences. Vol,**155**, pp. 286–294 (2019).
6. Le. Ch. Khanh. The Journal of the Acoustical Society of America. **109**(2): pp. 349–379 (2024).
7. R. Khalmuradov, Kh. Ismoilov, Frequency Analysis of Axisymmetric Vibrations of a Conical Shell in a Deformable Medium. AIP Conf. Proc. 3244, 060032, (2024). https://doi.org/10.1063/5.0241498
8. Kh. Khudoynazarov, K. Mamasoliyev, E. Ismoilov, Non-stationary influence of a transverse-isotropic cylindrical shell with a viscous compressed fluid. AIP Conference Proceedings. 3177, 050005, (2025). https://doi.org/10.1063/5.0294882
9. K. Mamasoliev, M. Mirsaidov, Mathematical model and analytical solution of the contact problem of bending a slab lying on an inhomogeneous combined base, AIP Conference Proceedings, 3177, 050006, (2025). https://doi.org/10.1063/5.0295291
10. Z. Khudoyberdiyev, Sh. Khudayberdiyeva, Sh. Yakhshiboyev, A. Begjanov, Numerical analysis of transverse forced vibration of a beam, AIP Conference Proceedings, 3177, 050010, (2025). https://doi.org/10.1063/5.02944897
11. Z. Khudoyberdiyev, Z. Suyunova, A. Begjanov, J. Khasanov, The longitudinal and transverse vibrations of a three-layered plate AIP Conf. Proc. 3177, 050012, (2025). https://doi.org/10.1063/5.029489
12. D. Kholikov, J. Abdurazzoqov, R. Usmonov, K. Xaydarova. Free torsional vibration of an elastic thin-walled cylindrical shell with variable cross section, AIP Conference Proceedings, 060029-1, (2024). https://doi.org/10.1063/5.0241748
13. K. Buranov, Q. Li, and V. L. Popov, Area and boundary contributions to friction in an adhesive contact of power-law indenters, AIP Conference Proceedings, 3177, 040002 (2025). https://doi.org/10.1063/5.0294715
14. O. Abdullayev, Kh. Ismoilov, A. Egamkulov, Application of the mixed finite element method in linear problems of elasticity theory. AIP Conference Proceedings, 3177, 070010 (2025). https://doi.org/10.1063/5.0295413
15. R. Khalmurodov, Kh. Ismoilov, Z. Khudayberdiyev, B. Babajanov, Vibration of a circular truncated conical shell interacting with a deformable medium, AIP Conference Proceedings 3177, 050011 (2025). https://doi.org/10.1063/5.0295160