The Nonstationary Longitudinal-Radial Vibrations of a Conical Shell

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Annotation: In this article, the problem of longitudinal–radial vibrations of a conical shell is investigated. Using the conducted research, the stressed and deformed states at the points of the conical shell's cross-section are studied in detail. Conical shell elements are widely used in mechanical engineering, energy, and aerospace industries. To determine their reliability and service life, it is necessary to thoroughly study the state of deformations and stresses. Therefore, the elastic deformations of the conical shell, the internal stress forces arising during longitudinal–radial vibrations, the maximum stresses and their distribution at the points of the shell's cross-section, as well as the possibilities of reducing deformations and optimizing the structure, are analyzed. The results of the study can be applied to improve the reliability of conical shell elements in industrial structures, optimize production processes, and determine the selection of materials.

Keywords: shell, boundary condition, longitudinal-radial vibration, deformation, displacement, stress.

INTRODUCTION.

Structural elements in the form of conical shells are an integral part of modern industrial technologies, and their reliability and efficiency depend on many factors. Such conical shell structures often operate under very high pressure and strong vibrations. These conditions require a deeper study of their stressed and deformed states. In particular, the longitudinal–radial vibrations of conical shells often lead to stress concentration at points of their cross-sections. This, in turn, can cause material stretching and various types of failures. When analyzing the longitudinal–radial vibrations of conical shells, their geometric dimensions, the physical properties of the materials, and the boundary conditions on their surfaces play an important role. For example, in studies [1–3], the torsional vibrations of elastic conical shells made of composite materials were investigated. A new mathematical model has been developed [4] that takes into account the external forces acting on the outer surface of the conical shell. In addition, using the finite element method, the vibrations of conical shell elements together with structural components were analyzed, and the effects of the conical shell's geometric parameters and boundary conditions on vibration frequencies were studied.

When determining the deformed states, it is necessary to evaluate the stresses at points of the shell's cross-section. In such cases, the classical small deformation theory is applied to determine the material's elastic properties and its deformed state. At the same time, the possibilities of using new materials and technologies in analyzing the deformed states of conical shell elements are also being explored. For instance, the nonlinear vibration characteristics of conical shells made from functionally graded materials reinforced with graphene nanoplatelets have been studied, and the effects of material porosity, graphene distribution, and the elastic foundation have been analyzed [5-10].

In this article, the longitudinal and radial vibrations of a conical shell and its stressed–deformed states at the points of the cross-section are analyzed mathematically. Based on numerical experiments conducted using the Finite Element Method (FEM), the distribution of deformations and stresses is determined. The results of the study help to identify the parameters necessary for improving the reliability and extending the service life of conical shell structures.

STATEMENT OF THE PROBLEMS (ISSUE)

We take an infinitesimal element of length l from an infinitely long, elastic, homogeneous, isotropic conical shell. A cylindrical coordinate system $Or\theta z$ is placed at the center of the small cross-section of this element (Fig. 1). In this case, the Oz-axis is directed along the axis of the conical shell. The Or-axis is directed along the radius of the conical shell's cross-section [11-15]. The inner radius of the conical shell at section $z=0$ is denoted by r_0. The thickness of the conical shell is d. The generator of the conical shell forms an angle of α with its axis. The inner radius of the conical shell at section $z=l$ is denoted by r_1, and the outer radius by r_2. Then, the following relationship holds:

$$r_1 = r_0 + z \cdot tg\alpha \; ; \; r_2 = r_0 + z \cdot tg\alpha + d$$

For the material of the conical shell, the Lamé coefficients are λ and μ. The density of the shell material is ρ.

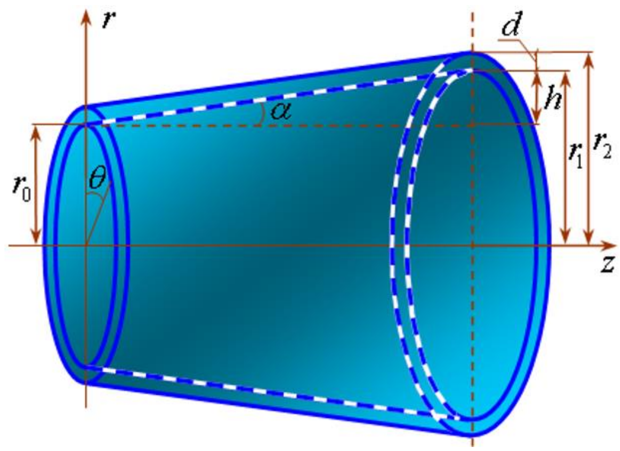

FIGURE 1. Conical shell

We assume the problem of longitudinal–radial vibrations of the conical shell to be axisymmetric. Therefore, among the components of the displacement vector of points in the cross-section of the conical shell, U_r and U_z are nonzero; among the components of the stress tensor, σ_{rr}, σ_{zz}, $\sigma_{\theta\theta}$ and τ_{rz} are nonzero; and among the components of the strain tensor, ε_{rr}, ε_{zz}, $\varepsilon_{\theta\theta}$ and γ_{zr} are nonzero. In the $Or\theta z$ cylindrical coordinate system, the system of differential equations of motion for the points of the conical shell, expressed in terms of stresses, can be written as follows:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \rho \frac{\partial^2 U_r}{\partial t^2}; \quad \frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\tau_{rz}}{r} = \rho \frac{\partial^2 U_z}{\partial t^2}. \tag{1}$$

For the points of the conical shell, the relationships between stresses and strains are expressed in the following form [15]:

$$\sigma_{rr} = A_{11} \frac{\partial U_r}{\partial r} + A_{12} \frac{U_r}{r} + A_{13} \frac{\partial U_z}{\partial z}; \quad \sigma_{\theta\theta} = A_{12} \frac{\partial U_r}{\partial r} + A_{11} \frac{U_r}{r} + A_{13} \frac{\partial U_z}{\partial z};$$
$$\sigma_{zz} = A_{13} \left(\frac{\partial U_r}{\partial r} + \frac{U_r}{r} \right) + A_{33} \frac{\partial U_z}{\partial z}; \quad \tau_{rz} = A_{44} \left(\frac{\partial U_r}{\partial z} + \frac{\partial U_z}{\partial r} \right). \tag{2}$$

Here, A_{ij} - are the elastic constants.

By substituting expressions (2), written for the points of the conical shell's cross-section during longitudinal–radial vibrations, into the previously derived system of motion equations (1), we can obtain the following system of equations with respect to the longitudinal and radial displacements of the conical shell points:

$$A_{11}\left(\frac{\partial^2 U_r}{\partial r^2}+\frac{1}{r}\frac{\partial U_r}{\partial r}-\frac{1}{r^2}U_r\right)+A_{44}\left(\frac{\partial^2 U_r}{\partial z^2}\right)+\left(A_{13}+A_{44}\right)\frac{\partial^2 U_r}{\partial r\partial z}=\rho\frac{\partial^2 U_r}{\partial t^2};$$
$$A_{44}\left(\frac{\partial^2 U_z}{\partial r^2}+\frac{1}{r}\frac{\partial U_z}{\partial r}\right)+A_{33}\frac{\partial^2 U_z}{\partial z^2}+\left(A_{13}+A_{44}\right)\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\frac{\partial U_r}{\partial z}=\rho\frac{\partial^2 U_z}{\partial t^2}. \tag{3}$$

The longitudinal–radial vibrations of the conical shell occur under the action of internal and external dynamic forces. In this case, the following boundary conditions are applicable on the $r=r_1$ and $r=r_2$ surfaces of the conical shell.

$$\sigma_{rr}(r,z,t)\big|_{r=r_i}=f_{ri}(z,t);\ \ \tau_{rz}(r,z,t)\big|_{r=r_i}=f_{zi}(z,t);\ i=1,2. \tag{4}$$

Thus, solving the problem of longitudinal–radial vibrations of the conical shell reduces to solving the system of equations (3) with the boundary conditions (4) and zero initial conditions.

SOLUTION OF THE PROBLEM

Using Laplace integral transforms, we express the system of equations (4) in terms of the $\tilde{U}_r$ and $\tilde{U}_z$ components of the displacement vector. By simplifying the resulting system of equations, we introduce the following notations:

$$\tilde{\Delta}_0=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2};\ \ \tilde{B}_1=\frac{1}{\tilde{A}_{11}}\left(\tilde{A}_{13}+\tilde{A}_{44}\right);\ \ \tilde{B}_2=\frac{1}{\tilde{A}_{44}}\left(\tilde{A}_{13}+\tilde{A}_{44}\right); \tag{5}$$

Taking into account the notations in (5), we differentiate the second equation of the system (4) with respect to the r variable and write it in the following form:

$$\tilde{\Delta}_0\tilde{U}_r+\left(\frac{\tilde{C}_{44}}{\tilde{C}_{11}}k^2-\frac{\rho}{\tilde{C}_{11}}p^2\right)\tilde{U}_r-k\tilde{B}_1\frac{\partial\tilde{U}_z}{\partial r}=0;$$
$$\tilde{\Delta}_0\frac{\partial\tilde{U}_z}{\partial r}-\left(\frac{\tilde{C}_{33}}{\tilde{C}_{44}}k^2+\frac{\rho}{\tilde{C}_{44}}p^2\right)\frac{\partial\tilde{U}_z}{\partial r}-k\tilde{B}_2\tilde{\Delta}_0\tilde{U}_r=0; \tag{6}$$

By solving the system of equations (6), we obtain the following solutions:

$$\tilde{U}_r=A_1I_1\left(\alpha_1 r\right)+D_1K_1\left(\alpha_1 r\right)+A_2I_1\left(\alpha_2 r\right)+D_2K_1\left(\alpha_2 r\right); \tag{7}$$

$$k\tilde{B}_1\tilde{U}_z=\frac{\alpha_1^2-\alpha^2}{\alpha_1}\left[A_1I_0\left(\alpha_1 r\right)+D_1K_0\left(\alpha_1 r\right)\right]+\frac{\alpha_2^2-\alpha^2}{\alpha_1}\left[A_2I_0\left(\alpha_2 r\right)+D_2K_0\left(\alpha_2 r\right)\right]; \tag{8}$$

Here, α,α_1,α_2 are coefficients that depend on the $\tilde{A}_{ij}$ elastic constants of the material. The displacement expressions (7) and (8) are expanded into power series with respect to the radial coordinate r. Then, denoting them as $r=\xi$ and $n=0$, we substitute the leading terms of the displacements $\tilde{U}_{r,0}$, $\tilde{U}_{r,1}$, $\tilde{U}_{z,0}$ and $\tilde{U}_{z,1}$ on the surface with radius ξ.

We replace the σ_{rr}, σ_{rz} stresses and f_{rr}, f_{rz} external forces in the boundary conditions (4) in the same way as the displacements. The resulting expressions for the substituted stresses are then applied to the boundary conditions (4), and can be written as follows:

$$\tilde{A}_{11}\frac{\partial\tilde{U}_r}{\partial r}+\tilde{A}_{12}\frac{\tilde{U}_r}{r}-k\tilde{A}_{13}\tilde{U}_z=f_{ri}(k,t);\ \ \tilde{A}_{44}\left(k\tilde{U}_r+\frac{\partial\tilde{U}_z}{\partial r}\right)=f_{rzi}(k,t);\ i=1,2. \tag{9}$$

We substitute the expressions (7) and (8) into this system of equations (9):

$$\tilde{C}_{11}\left\{\alpha_1\left[A_1I_0\left(\alpha_1r_2\right)-D_1K_0\left(\alpha_1r_2\right)\right]+\alpha_2\left[A_2I_0\left(\alpha_2r_2\right)-D_1K_0\left(\alpha_2r_2\right)\right]\right\}+$$
$$+\frac{\tilde{C}_{12}}{r}\left[A_1I_1\left(\alpha_1r_2\right)+D_1K_1\left(\alpha_1r_2\right)+A_2I_1\left(\alpha_2r_2\right)+D_2K_1\left(\alpha_2r_2\right)\right]-$$
$$-k\tilde{C}_{13}\left\{\frac{\alpha_1^2-\alpha^2}{\alpha_1k\tilde{B}_1}\left[A_1I_0\left(\alpha_1r_2\right)-D_1K_0\left(\alpha_1r_2\right)\right]+\frac{\alpha_2^2-\alpha^2}{\alpha_2k\tilde{B}_1}\left[A_2I_0\left(\alpha_2r_2\right)-D_2K_0\left(\alpha_2r_2\right)\right]\right\}=\tilde{f}_r^{(2)}; \tag{10}$$

$$\left(\alpha_1^2-\alpha^2+k^2\tilde{B}_1\right)\left[A_1I_1\left(\alpha_1r_2\right)+D_1K_1\left(\alpha_1r_2\right)\right]+\left(\alpha_2^2-\alpha^2+k^2\tilde{B}_1\right)\left[A_2I_1\left(\alpha_2r_2\right)+\right.$$
$$\left.+D_2K_1\left(\alpha_2r_2\right)\right]=\tilde{B}_1\tilde{C}_{44}^{-1}\left[k\tilde{f}_{rz}^{(2)}\right];$$

In the final system of equations (10), we expand the $I_i\left(\alpha_1r_2\right),\ K_i\left(\alpha_1r_2\right),\ I_i\left(\alpha_2r_2\right),\ K_i\left(\alpha_2r_2\right)$ Bessel functions in terms of r_2, and the $I_i\left(\alpha_1r_1\right),\ K_i\left(\alpha_1r_1\right),\ I_i\left(\alpha_2r_1\right),\ K_i\left(\alpha_2r_1\right),\ (i=0,1)$ Bessel functions in terms of r_1. By substituting the values of the constants into the resulting expansions, we obtain a system of four algebraic equations for the leading terms of the transformed displacements $\tilde{U}_{r,0},\tilde{U}_{z,0},\tilde{U}_{r,1},\tilde{U}_{z,1}$.

By applying operators to the system of equations (10) and performing mathematical simplifications, we obtain the following system of equations:

$$\begin{aligned} &N_{j1}U_{z,0}+M_{j1}U_{r,0}+K_{j1}U_{z,1}+L_{j1}U_{r,1}=S_{j1}f_{ri}(z,t);\\ &N_{j2}U_{z,0}+M_{j2}U_{r,0}+K_{j2}U_{z,1}+L_{j2}U_{r,1}=S_{j2}f_{rzi}(z,t);\quad i=1,2;\ j=1,2; \end{aligned}\tag{11}$$

Here, $N_{j1},M_{j1},\ K_{j1},\ L_{j1},\ S_{j1}\ (j=\overline{1,4})$ are differential operators, and their form is as follows. For example,

$$N_{j1}=c_{j1}\frac{\partial^4}{\partial t^4}+c_{j2}\frac{\partial^4}{\partial t^2\partial z^2}+c_{j3}\frac{\partial^4}{\partial z^4}+c_{j4}\frac{\partial^2}{\partial t^2}+c_{j5}\frac{\partial^2}{\partial z^2},$$

$$M_{j1}=g_{j1}\frac{\partial^4}{\partial t^4}+g_{j2}\frac{\partial^4}{\partial t^2\partial z^2}+g_{j3}\frac{\partial^4}{\partial z^4}+g_{j4}\frac{\partial^3}{\partial t^2\partial z}+g_{j5}\frac{\partial^3}{\partial z^3}+g_{j6}\frac{\partial^2}{\partial t^2}+g_{j7}\frac{\partial^2}{\partial z^2},$$

CONCLUSION

By solving this system of differential equations (11), it is possible to determine the sought functions, such as $U_{r,0},U_{z,0},U_{r,1}$ and $U_{z,1}$. They are of significant importance in analyzing the longitudinal–radial vibrations of a circular truncated conical shell. Using these sought functions, it becomes possible to determine the displacements and stresses at any point of the cross-section of the circular conical shell.

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