Variational Method in Calculating the Resistance of Finite-Length Underground Pipelines to Seismic Impacts

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Abstract. The flow around a cylindrical rod embedded in a layer of elastic medium in which a plane wave propagates was studied. It is assumed that the cylinder, under the action of a distributed force on the surface and ends, performs only horizontal movement. To solve the problem, the variational method of V.Z. Vlasov and the Fourier method were used. The results are presented as a function of the displacement of the rod's middle cross-section relative to the moving soil over time for different values of Young's modulus, as well as in the form of longitudinal stress curves in the initial cross-section of the rod over time. It was found that the maximum stress values over time in the section are achieved after the complete passage of the wave along the length of the rod.

INTRODUCTION

As is known, the basis of the variational principle of mechanics is the assertion that in real processes some functionals have stationary values. These principles dictate a special structure of the equations of mechanics, reflecting the properties of reciprocity of physical effects: the action of one field on another generates a reverse, in a sense, symmetrical effect. The soils used in the dynamics of underground pipelines are a structurally heterogeneous layered continuous medium, the state of which is generally determined by nonlinear laws of deformation. The relationship between the deformation of layers located parallel to the pipeline axis and forming the structural composition of the medium under static conditions has been studied in sufficient detail in [1-6]. Based on the criterion of taking into account all the properties of the environment, a distinction is made between a model of general deformations (an example is the model of a linearly deformed half-space) and various models of local deformations, among which the most widely used is the Winkler (Winkler-Pasternak) model [3-5]. The main disadvantage of this model is that it does not have the ability to “distribute” the load, while experience shows that it does not come into contact with the underground structure, in particular with the pipeline. Soil deformation also occurs beyond the loaded area. This circumstance necessitates the refinement of foundation calculation models and the development of methods for calculating complex structures that take into account the spatial flexibility of the soil. In contrast to this model, based on the general variational principle, in works [6-8] a technical theory for calculating a structure on an elastic foundation is proposed, which is more accurate and at the same time simpler, which is very flexible and allows solving not only the basic problems of calculating beams and slabs on an elastic foundation.

METHODOLOGY

Based on the model of the medium, we consider an elastic cylindrical layer of the medium with a total thickness R, a deformable circular rod of radius a and length L concentrically embedded in it. We set the origin of the polar coordinate in the initial section of the rod at point 0 and direct the $0z$ axis along the axis of the rod, and the $0r$ axis perpendicular to it (Fig.1). We assume that the zone of rod penetration is determined by the region $0<r<a$, $0<z<L$. We denote by $u(r, z, t)$ and $v(r, z, t)$ the displacement of soil particles in the directions of the axes 0r and 0z in axisymmetric coordinates (r, z), t is time.

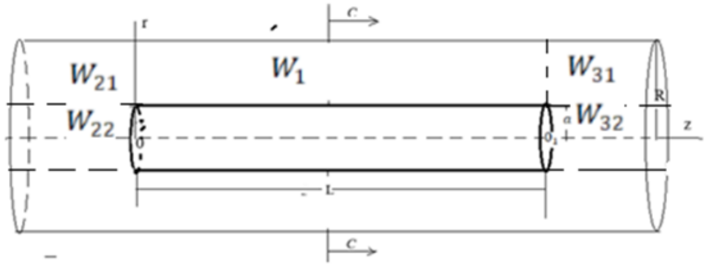


FIGURE 1. Diagram of longitudinal wave flow around a rod embedded in a layer of a medium

The components of the stress tensor are represented in the form.

$$\sigma_{zz} = \frac{E_0}{1-v_0^2}\left(\varepsilon_{zz} + v_0\varepsilon_{rr}\right), \tag{1}$$

$$\sigma_{rr} = \frac{E_0}{1-v_0^2}\left(v_0\varepsilon_{zz} + \varepsilon_{rr}\right), \tag{2}$$

$$\tau_{rz} = \tau_{zr} = \frac{E_0}{2(1+v_0)}\varepsilon_{rz}, \tag{3}$$

where

$$\varepsilon_{rr} = \frac{\partial u}{\partial r},\ \varepsilon_{zz} = \frac{\partial v}{\partial z},\ \varepsilon_{rz} = \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}, \tag{4}$$

E_0 and v_0 are Young's modulus and Poisson's ratio of the soil environment.

We assume that a plane wave propagates through the layer, behind the front of which the longitudinal displacement of particles of the medium v is given by the law

$$v_c = w_0\left(t - z/c_0\right)$$

here $c_0 = \sqrt{\frac{E_0}{\rho_0}}$ is the propagation speed of the longitudinal wave in the layer

We assume that the rod performs, under the action of a distributed force on the surface and ends, a predominantly horizontal movement, and in connection with this, we further adopt $u(r, z, t)\approx 0$. The displacement of layer particles $v(r, z, t)$ and the rod cross-section $v_1(z, t)$ relative to the moving soil particles is denoted by $W_1(z, t)$ and $w(r, z, t)$, respectively.

$$v_1(z,t) = W_1(z,t) + v_0\left(t - z/c_0\right) \tag{5}$$

here $c_0 = \sqrt{\frac{E_0}{\rho_0}}$

$$v(r,z,t) = w(r,z,t) + w_0\left(t - z/c_0\right) \tag{6}$$

We consider the layer to be an elastic medium consisting of three zones with known mechanical properties. The first is the region occupied by the rod $a\le r<R$, $0<z<L$. In this region, the movement occurs in the direction of the $0z$ axis and the layer thickness is selected, according to work [6], as a function of the variable r, in such a way that the movement of the layer particles is represented in the form

$$w = W_1(z,t)\varphi_1(r) \text{ for } 0<z<L,\ a\le r<R \tag{7}$$

The second zone $L<z<\infty$, $0\le r<R$ consists of two parts. In the first part $L<z<\infty$, $0\le r<a$ the displacement is represented as

$$w(r,z,t) = W_{22}(z,t)\varphi_2(r) \text{ for } L<z<\infty,\ 0\le r<a \tag{8}$$

In the second part we assume

$$w(r,z,t) = W_{21}(z,t)\varphi_1(r) \text{ for } L<z<\infty,\ a\le r<R \tag{9}$$

The third zone $-\infty<z<0$, $0\le r<R$ also consists of two parts, where the displacements are represented as

$$w(r,z,t)=W_{32}(z,t)\varphi_2(r) \text{ for } -\infty<z<0,\ 0\le r<a \tag{10}$$

$$w(r,z,t)=W_{31}(z,t)\varphi_1(r) \text{ for } -\infty<z<0,\ a\le r<R \tag{11}$$

Following the work [6], we compile the work of external normal forces for all zones in relation to the selected annular strip, where $\varphi(a)=1$ is taken

$$\begin{aligned}
&2\pi\int_a^R \frac{\partial\sigma_{zz}}{\partial z}\varphi_1(r)rdr-2\pi\int_a^R \tau_{rz}\varphi_1'(r)rdr-p(z,t)=0 \quad for \quad 0<z<L,\\
&2\pi\int_a^R \frac{\partial\sigma_{zz}}{\partial z}\varphi_1(r)rdr-2\pi\int_a^R \tau_{rz}\varphi_1'(r)rdr=0 \quad for \quad L<z<\infty,\ -\infty<z<0,\\
&2\pi\int_0^a \frac{\partial\sigma_{zz}}{\partial z}\varphi_2(r)rdr-2\pi\int_0^a \tau_{rz}\varphi_2'(r)rdr-q_1(z,t)=0 \quad for \quad L<z<\infty,\\
&2\pi\int_0^a \frac{\partial\sigma_{zz}}{\partial z}\varphi_2(r)rdr-2\pi\int_0^a \tau_{rz}\varphi_2'(r)rdr-q_2(z,t)=0 \quad for \quad -\infty<z<0
\end{aligned} \tag{12}$$

where $p(z, t)$ is the intensity of the tangential forces on the lateral surface of the rod, $q_1(z, t)$ and $q_2(z, t)$ are the distributed loads acting on the right and left ends

Using dependencies (1) and (3) taking into account (5) and (6) from (7) - (9) we obtain

$$2s_1\frac{\partial^2 W_1}{\partial z^2}-k_1W_1-m_1\frac{\partial^2 W_1}{\partial t^2}-p(z,t)=(m_1-s_{01})\ddot{v}_0\left(t-\frac{z}{c_0}\right) \text{ for } 0<z<L,\ 0\le r<a \tag{13}$$

$$2s_1\frac{\partial^2 W_{21}}{\partial z^2}-k_1W_{21}-m_1\frac{\partial^2 W_{21}}{\partial t^2}=(m_1-s_{01})\ddot{v}_0\left(t-\frac{z}{c_0}\right) \text{ for } L<z<\infty,\ a\le r<R \tag{14}$$

$$2s_2\frac{\partial^2 W_{22}}{\partial z^2}-k_2W_{22}-m_2\frac{\partial^2 W_{22}}{\partial t^2}-2\pi a q_1(z,t)=(m_2-s_{02})\ddot{v}_0\left(t-\frac{z}{c_0}\right) \text{ for } L<z<\infty,\ 0\le r<a \tag{15}$$

$$2s_1\frac{\partial^2 W_{31}}{\partial z^2}-k_1W_{31}-m_1\frac{\partial^2 W_{31}}{\partial t^2}=(m_1-s_{01})\ddot{v}_0\left(t-\frac{z}{c_0}\right) \text{ for } -\infty<z<0,\ a\le r<R \tag{16}$$

$$2s_2\frac{\partial^2 W_{32}}{\partial z^2}-k_2W_{32}-m_2\frac{\partial^2 W_{32}}{\partial t^2}+2\pi a q_2(z,t)=(m_2-s_{02})\ddot{v}_0\left(t-\frac{z}{c_0}\right) \text{ for } -\infty<z<0,\ 0\le r<a \tag{17}$$

where $s_1=\frac{2\pi E_0}{4(1+\nu_0)}\int_a^R \varphi_1^2(r)rdr$, $k_1=\frac{2\pi E_0}{4(1-\nu_0^2)}\int_a^R \varphi_1'^2(r)rdr$, $m_1=2\pi\rho_0\int_a^R \varphi_1^2(r)rdr$, $s_{01}=\frac{2s_1}{c_0^2}$,

$$s_2=\frac{2\pi E_0}{4(1+\nu_0)}\int_0^a \varphi_2^2(r)rdr,\quad k_2=\frac{2\pi E_0}{(1-\nu_0^2)}\int_0^a \varphi_2'^2(r)rdr,\quad m_2=2\pi\rho_0\int_0^a \varphi_2^2(r)rdr,\quad s_{02}=\frac{2s_2}{c_0^2}$$

We write the equation of motion of the rod in the form

$$EF\frac{\partial^2 W_1}{\partial z^2}-\rho F\frac{\partial^2 W_1}{\partial t^2}+p(z,t)=0$$

Substituting the expression $p(z, t)$ from (13) into the last equation, we obtain

$$(EF+2s_1)\frac{\partial^2 W_1}{\partial z^2}-(\rho F+m_1)\frac{\partial^2 W_1}{\partial t^2}-k_1W_1=(m_1-s_{01})\ddot{v}_0\left(t-\frac{z}{c_0}\right) \tag{18}$$

Following the work [6], in the zones $L<z<\infty$ $0\le r<a$ and $-\infty<z<0$, $0\le r<a$ we assume

$$W_{22}=W_{00}(t)=W_1(0,t),\quad W_{32}=W_{01}(t)=W_1(L,t) \tag{19}$$

where $W_{00}(t)$ and $W_{01}(t)$ are the displacements of the ends of the rod

Then from (12) and (14) we determine the loads $q_1(z, t)$ and $q_2(z, t)$

$$q_1(t)=-\frac{1}{2\pi a}\left[k_2W_{00}+m_2\ddot{W}_{00}+\left(m_2-\frac{2s_2}{c_0^2}\right)\ddot{v}_0(t)\right]$$

$$q_2(t) = -\frac{1}{2\pi a}\left[k_2 W_{01} + m_2 \ddot{W}_{01} + \left(m_2 - \frac{2s_2}{c_0^2}\right)\ddot{v}_0\left(t - \frac{L}{c_0}\right)\right]$$

Along the contours of the ends $z=0$, $z=L$ of the rod, fictitious forces Q_1 and Q_2 act with intensities

$$Q_1 = \frac{s_1}{\pi a}\frac{\partial W_{22}(0,t)}{\partial z}, \quad Q_2 = \frac{s_1}{\pi a}\frac{\partial W_{32}(L,t)}{\partial z}$$

Based on the known expressions for the loads $q_1(z, t)$, $q_2(z, t)$, Q_1 and Q_2, it is possible to formulate boundary conditions at the ends of the rod.

$$EF\frac{\partial W_1}{\partial z} = -\pi a^2 q_1(z,t) - 2\pi a Q_1 = -ak_2 W_{00} + 2s_1\frac{\partial W_{21}(L,t)}{\partial z} + am_2\ddot{W}_{00} - a(m_2 - s_{02})\ddot{v}_0(t) \text{ for } z=0 \tag{20}$$

$$EF\frac{\partial W_1}{\partial z} = \pi a^2 q_2(z,t) + 2\pi a Q_2 = -ak_2 W_{01} + 2s_1\frac{\partial W_{31}(L,t)}{\partial z} + am_2\ddot{W}_{01} - a(m_2 - s_{02})\ddot{v}_0\left(t - \frac{L}{c_0}\right) \text{ for } z=0 \tag{21}$$

Boundary conditions (20) and (21) contain derivatives of functions W_{21} and W_{31} that satisfy equations (14) and (16), thus, to determine the solution of equation (18), one must first determine the solutions of equations (14) and (16) that satisfy the conditions

$$W_{21} = W_{00}(t) \text{ for } z=L$$
$$W_{31} = W_{01}(t) \text{ for } z=0$$
$$W_{21} \to 0 \text{ for } z\to\infty$$
$$W_{31} \to 0 \text{ for } z\to-\infty$$

Let us consider a rigid contact of the ends of the rod with the layer. In this case, the displacement of the ends relative to the soil will be zero, i.e., it should be assumed that $W_{00}=W_{01}=0$. Then the functions $q_1(z, t)$ and $q_2(z, t)$ are determined by the formulas

$$q_1(t) = \frac{1}{2\pi a}(m_2 - s_{02})\ddot{v}_0(t)$$

$$q_2(t) = -\frac{1}{2\pi a}(m_2 - s_{02})\ddot{v}_0\left(t - \frac{L}{c_0}\right)$$

In this case, equations (14), (16) and (18) will not be related through boundary conditions (20) and (21) and will be integrated under the conditions

$$W_1 = 0 \text{ for } z=0,\ z=L$$
$$W_{21} = 0 \text{ for } z=L,\ W_{31} = 0 \text{ for } z=0$$
$$W_{21} \to 0 \text{ for } z\to\infty,\ W_{31} \to 0 \text{ for } z\to-\infty$$

Conditions (20) and (21) are written as

$$EF\frac{\partial W_1}{\partial z} = 2s_1\frac{\partial W_{21}(L,t)}{\partial z} - a(m_2 - s_{02})\ddot{v}_0(t) \tag{22}$$

$$EF\frac{\partial W_1}{\partial z} = 2s_1\frac{\partial W_{31}(L,t)}{\partial z} - a(m_2 - s_{02})\ddot{v}_0\left(t - \frac{L}{c_0}\right) \tag{23}$$

Assuming $m_1/2s_1\approx 0$, equations (14) and (16) are written as

$$2s_1\frac{\partial^2 W_{21}}{\partial z^2} - k_1 W_{21} = -s_{01}\ddot{v}_0\left(t - \frac{z}{c}\right) \text{ for } L<z<\infty \tag{25}$$

$$2s_1\frac{\partial^2 W_{31}}{\partial z^2} - k_1 W_{31} = -s_{01}\ddot{v}_0\left(t - \frac{z}{c}\right) \text{ for } -\infty<z<0 \tag{26}$$

with boundary conditions

$$W_{21}(L,t) = 0,\ W_{31}(0,t) = 0 \tag{27}$$

$$W_{21}(\infty,t) = 0,\ W_{31}(-\infty,t) = 0 \tag{28}$$

We will obtain the solution to equation (18) using the Fourier method [9], according to which we will represent it in the form of an expansion

$$W_1 = \sum_{n=1}^{\infty} T_n(t)\sin\frac{\pi n z}{L}$$

T_n satisfies the equations (derivative with respect to the variable (τ))

$$\ddot{T}_n + \bar{\mu}_n^2 T_n = 2A_0 \frac{M^2-1}{M^2} \int_0^{\tau/M} J(\tau - M\xi)\sin \pi n\xi d\xi \text{ for } \tau<M$$

$$\ddot{T}_n + \bar{\mu}_n^2 T_n = 2A_0 \frac{M^2-1}{M^2} \int_0^{1} J(\tau - M\xi)\sin \pi n\xi d\xi \text{ for } \tau>M$$

where $\xi = \frac{z}{L}$, $\tau = \frac{ct}{L}$, $k = \frac{k_1 L^2}{EF + 2s_1}$, $M = \frac{c_0}{c}$, $c = \sqrt{\frac{EF + 2s_1}{\rho F + m_1}}$, $\bar{\mu}_n = \sqrt{k + n^2\pi^2}$

$J(u) = \ddot{v}_0$ dimensionless function of the acceleration of particles of the soil medium behind the wave front, A_0 is a given value of the dimension of length.

The solutions of the last equations for T_n with zero initial conditions can be represented as

$$T_n = 2A_0 \frac{M^2-1}{M^2 \bar{\mu}_n} A_0 \int_0^{\tau/M} F_n(\tau)\sin[\bar{\mu}_n(t-\tau)]d\tau$$

Here $F_n(\tau) = \int_0^{\tau/M} J(\tau - M\xi)\sin \pi n\xi d\xi$ for $\tau<M$

$$F_n(\tau) = \int_0^{1} J(\tau - M\xi)\sin \pi n\xi d\xi \text{ for } \tau>M$$

The calculations were carried out by selecting the function $\varphi(r)$ satisfying the conditions $\varphi(a)$=1 and $\varphi(R)$=0 in the form

$$\varphi(r) = C_1 J_0(\lambda r) + C_2 K_0(\lambda r)$$

$$C_1 = -\frac{K_0(\lambda R)}{J_0(\lambda a)K_0(\lambda R) - J_0(\lambda R)K_0(\lambda a)},\ C_2 = \frac{J_0(\lambda R)}{J_0(\lambda a)K_0(\lambda R) - J_0(\lambda R)K_0(\lambda a)},$$

where $J_0(z)$ and $K_0(z)$ are Bessel functions of the second kind of zero order.

The change in the displacement of soil particles behind the wave front is taken according to the law

$$v_0 = \tau \sin \omega_0 \tau \exp(-\alpha_0 \tau)$$

where the dimensionless quantities ω_0 and α_0 are expressed through the frequency ω_{00}, the length of the rod L and the coefficient of friction according to the formulas

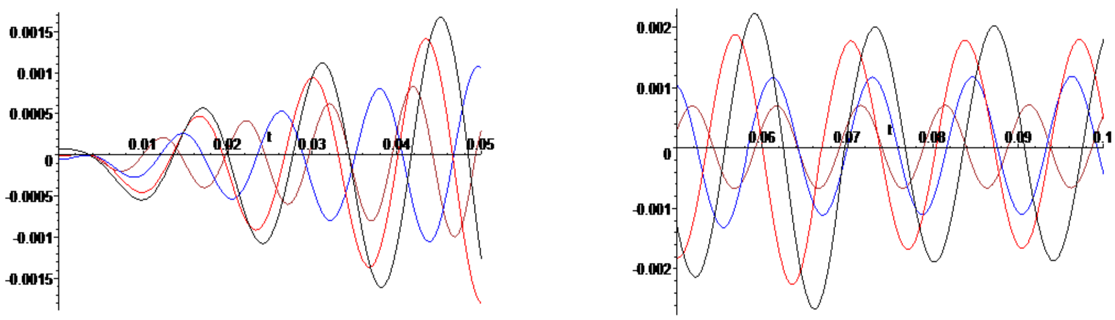
$$\omega_0 = \omega_{00} L / c,\ \alpha_0 = \alpha_{00} L / c$$

RESULTS

The calculations were performed for different values of Young's modulus E_0 of the medium, with the initial data E=4·10^{4}MPa, ρ=5000kg/m^3, ρ_0=1700kg/m^3, R=1m, v_0=0.3, a=0.2m, L=10m, A_0=0.0005m, ω_0=5(1/s), α_{00}=1(1/s).

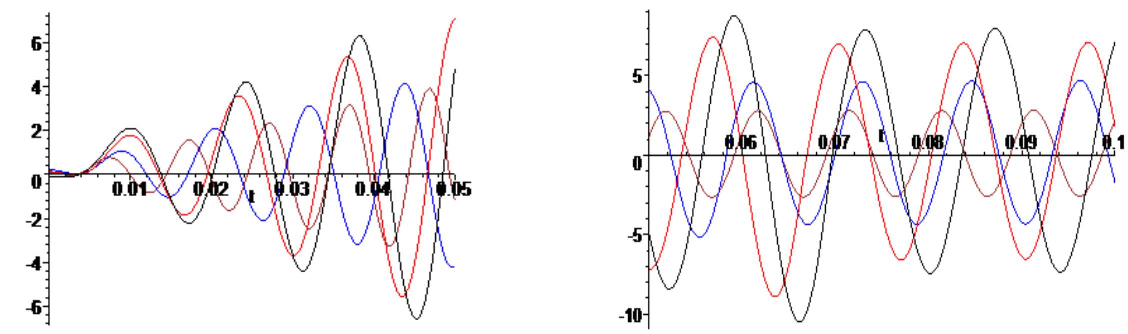
The results of calculations of the dependence of the displacement $W_1(0.5L, t)$ (m) of the average cross-section of the rod relative to the moving soil on time t(sec) for different values of Young's modulus E_0 are presented in Fig. 2 (a. b), where Fig. 2a shows the changes in the displacement of the cross-section for moments in time when the wave front flows around the full length of the rod. In this case, the fronts of the wave propagating in the rod and soil reach the end section in 0.017 sec and 0.35 sec, respectively. It is evident that the maximum displacement values in the cross-section of the rod decrease with increasing Young's modulus and reach their maximum after the rod is completely flown around by a wave propagating in the soil.

Fig. 3 shows the curves of the dependence of the longitudinal stress σ (MPa) in the initial section of the rod on time t (sec). From the analysis of the curves it follows that the maximum values of stress over time in the section are achieved after the complete passage of the wave along the length of the rod. In this case, during the interaction of the rod with the environment (soil), the change in stress in the rod over time is oscillatory in nature and for the selected time intervals, the amplitude of oscillations decreases with an increase in the Young's modulus of the environment.



a) b)

FIGURE 2. Curves of the dependence of the displacement relative to the ground of the average cross-section of the rod $W_1(0.5L, t)$ on time t(sec) for different values of the Young's modulus of the soil medium E_0 (MPa):
1) E_0=35, 2) E_0=45, 3) E_0=50, 4) E_0=70



a) b)

FIGURE 3. Curves of the dependence of the stress relative to the ground of the average cross-section of the rod $W_1(0.5L, t)$ on time t(sec) for different values of the Young's modulus of the soil medium E_0 (MPa):
1) E_0=35, 2) E_0=45, 3) E_0=50, 4) E_0=70

CONCLUSION

The process of flow around a cylindrical rod embedded in a layer of elastic medium is considered. To solve the problem, the variational method of V.Z. Vlasov and the Fourier method were used. In this case, wave equations of motion were obtained for the horizontal displacement of the rod and the soil environment surrounding it. It was established that the fronts of the wave propagating in the rod and soil reach the end section in 0.35 sec and 0.0176 sec, respectively. From the analysis of the longitudinal stress curves in the initial cross-section of the rod over time, it was found that the maximum stress values over time in the cross-section are achieved after the complete passage of the wave along the length of the rod.

ACKNOWLEDGMENTS

The research in this article was carried out with budgetary funding from the Institute of Mechanics and Earthquake Resistance of Structures named after M.T. Urazbaev of the Academy of Sciences of the Republic of Uzbekistan.

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