Dependence of Stress Concentration on Crack Size

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Abstract. Many structural elements and machine parts contain initial or acquired stress concentrators, such as cracks and grooves. While some of these features may appear due to technological requirements, most arise during operation. A critical aspect of assessing the strength state is selecting the method for evaluating limit states based on the applicable strength criteria. These criteria depend on several factors, with the most significant being the material properties, the loading sequence and type, and the geometric dimensions of the stress concentrator. This article explores the relationship between stress growth and the size of the initial crack. Understanding this relationship has practical implications for evaluating the strength and load-bearing capacity of structures or materials that exhibit identical cracks of varying modulus values. This assessment is also vital for engineering evaluations of material strength in the presence of crack systems.

INTRODUCTION

Analytical methods can predict solutions for a limited class of problems involving linear single cracks. Among the most promising numerical techniques are the boundary element methods, which are capable of accounting for stresses at infinity, as well as the full set of methods utilizing complex variable functions. By employing these methods, it becomes possible to address problems that involve singularities.

In [1], a displacement discontinuity method for modeling axisymmetric cracks in an elastic half-space or full space is described. The formulation is based on hypersingular integral equations that relate discontinuities in displacement and tension along the crack. The crack is discretized into constant-strength displacement discontinuity elements, where each element represents a cone section. The accuracy of the solution at the crack tip is ensured by adding corrective stresses along the tip element. The method is validated by comparison with analytical and numerical benchmark solutions.

The initiation of a longitudinal shear crack (mode III failure) in elastoplastic materials under ultimate strain is considered in [2]. The crack propagation criterion is formulated based on a modified Leonov–Panasyuk–Dugdale model using an additional parameter—the width of the plastic zone (the width of the pre-fracture zone). A two-parameter criterion for quasi-brittle fracture of an elastoplastic material for mode III cracks is formulated. The proposed failure criterion includes a strain criterion formulated at the crack tip and a force criterion formulated at the tip of a model crack. For quasi-viscous and ductile failure modes, the ultimate loads are determined numerically. The propagation of plastic zones in the vicinity of the crack tip under quasi-static loading is consistently described by the nonlinear finite element method.

Reference [3] presents a straightforward and systematic approach for implementing the boundary element method to solve the Laplace equation using the MATLAB software package. Reference [4] explores the application of numerical methods based on the discontinuous displacement method for modeling the growth of curvilinear cracks under complex loading conditions and interactions, comparing the results obtained with experimental data. References [5] and [6] discuss algorithms for modeling failure in an elastic two-dimensional medium where stress concentrators are present under tensile loads, as well as methods for strengthening materials featuring central cracks.

MATERIALS, METHODS, AND OBJECTS OF STUDY

When addressing issues involving one or more narrow slit-like notches and cracks, it is appropriate to employ the boundary element method using discontinuous displacements. This approach relies on the analytical solution for an infinite plane subjected to constant rupture. The method of discontinuous displacements is founded on the idea that continuous displacement discontinuities along the crack can be approximated by discrete values. In this approach, only cracks that are normal to rupture and shear are considered. Unfortunately, this method does not accommodate anti-plane deformation.

To solve the elastic problem for a single crack of types I and II, we derive displacements (the fundamental solution) based on the Papkovich-Neuber solution [7]. The equilibrium equation has the following form:

$$\begin{array}{l} \dfrac{\partial \sigma_{xx}}{\partial x}+\dfrac{\partial \sigma_{xy}}{\partial y}+X=0 \\ \dfrac{\partial \sigma_{xy}}{\partial x}+\dfrac{\partial \sigma_{yy}}{\partial y}+Y=0 \end{array}. \quad (1)$$

Under plane deformation conditions, the components of displacement along the *x*- and *y*-axes in a homogeneous, isotropic, linearly elastic body are defined as:

$$\left.\begin{array}{l} u_x = B_x - \dfrac{1}{4(1-\nu)}\dfrac{\partial}{\partial x}(xB_x + yB_y + \beta), \\ u_y = B_y - \dfrac{1}{4(1-\nu)}\dfrac{\partial}{\partial y}(xB_x + yB_y + \beta), \end{array}\right\}, \quad (2)$$

where ν - is Poisson's ratio, and B_x, B_y, and β - are Papkovich functions, which, in the absence of volume forces, satisfy the following Laplace equation:

$$\nabla^2 B_x = 0,\ \nabla^2 B_y = 0,\ \nabla^2 \beta = 0. \quad (3)$$

It will be appropriate to choose two partial forms of the Papkovich set function, one set corresponding to the body with plane $y = 0$ free of transverse load, and the other set - to the body with the same plane free of normal forces. These two particular solutions are found by substituting equations (2) into the plane strain stress-strain relations and choosing B_x, B_y , and β such that either $\sigma_{xy} = 0$, or $\sigma_{yy} = 0$ for $y=0$.

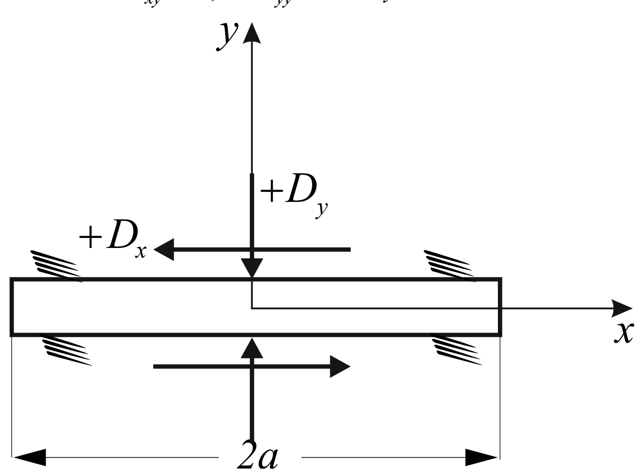


FIGURE 1. Components of a constant displacement discontinuity

For constant elements (assumed constant for each element), the discontinuous displacement method has the following form [8,9]:

$$
\begin{aligned}
u_x &= D_x\left[2(1-\nu)f_{,y} - yf_{,xx}\right] + D_y\left[-(1-2\nu)f_{,x} - yf_{,xy}\right],\\
u_y &= D_x\left[(1-2\nu)f_{,x} - yf_{,xy}\right] + D_y\left[2(1-\nu)f_{,y} - yf_{,yy}\right],\\
\sigma_{xx} &= 2GD_x\left[2f_{,xy} + yf_{,xyy}\right] + 2GD_y\left[f_{,yy} + yf_{,yyy}\right],\\
\sigma_{yy} &= 2GD_x\left[-yf_{,xyy}\right] + 2GD_y\left[f_{,yy} - yf_{,yyy}\right],\\
\sigma_{xy} &= 2GD_x\left[f_{,yy} + yf_{,yyy}\right] + 2GD_y\left[-yf_{,xyy}\right],
\end{aligned}
\tag{4}
$$

where D_i – are the discontinuities of displacements, and

$$
\begin{aligned}
f(x,y) = -\frac{1}{4\pi(1-\nu)}\Bigg[y\left(\arctan\left(\frac{y}{x+a}\right) - \arctan\left(\frac{y}{x-ay}\right)\right) + \\
(x-a)\ln\sqrt{(x-a)^2+y^2} - (x+a)\ln\sqrt{(x+a)^2+y^2}\Bigg].
\end{aligned}
\tag{5}
$$

Since the orientation of the cracks in the plane can be arbitrary, it is more convenient to use a local coordinate system on each boundary element. In this case, only normal stress *n* (normal to the boundary) and shear stress *s* will be determined.

The discontinuous displacement method (DDM) for two-dimensional problems with cracks was proposed by Crouch [8]. In this method, the crack is approximated by segments with a constant discontinuity of displacements *D* along each segment. Using the exact solution for such a segment, stresses σ at the center of the *j*-th crack element are expressed through the discontinuities of displacements *D* as a superposition of solutions from all *i*-th elements:

$$
\begin{aligned}
\sigma_s^i &= \sum_{j=1}^{N} A_{ss}^{ij} D_s^j + \sum_{j=1}^{N} A_{sn}^{ij} D_n^j\\
\sigma_n^i &= \sum_{j=1}^{N} A_{ns}^{ij} D_s^j + \sum_{j=1}^{N} A_{nn}^{ij} D_n^j
\end{aligned},
\tag{6}
$$

where subscripts s and n denote the shear and normal components of vectors σ and *D*; functions A_{ss}, A_{sn}, A_{ns}, A_{nn} are exact solutions for a constant discontinuity of displacements on a segment obtained using the Papkovich functions. A system of linear algebraic equations was formed, and unknowns *D* are found from its solution. The problem was solved using the developed software package, and the system of linear algebraic equations was solved using the Gauss method [10].

STATEMENT OF THE PROBLEM

We consider the problem (Fig. 2) of loading a plate (plane stress state). The quantities are specified in dimensionless abstract quantities (P=1, E=2, ν=0.27). We vary the values of y_{ij}. We assume that the crack is located on the symmetry axis of the body (x=5).

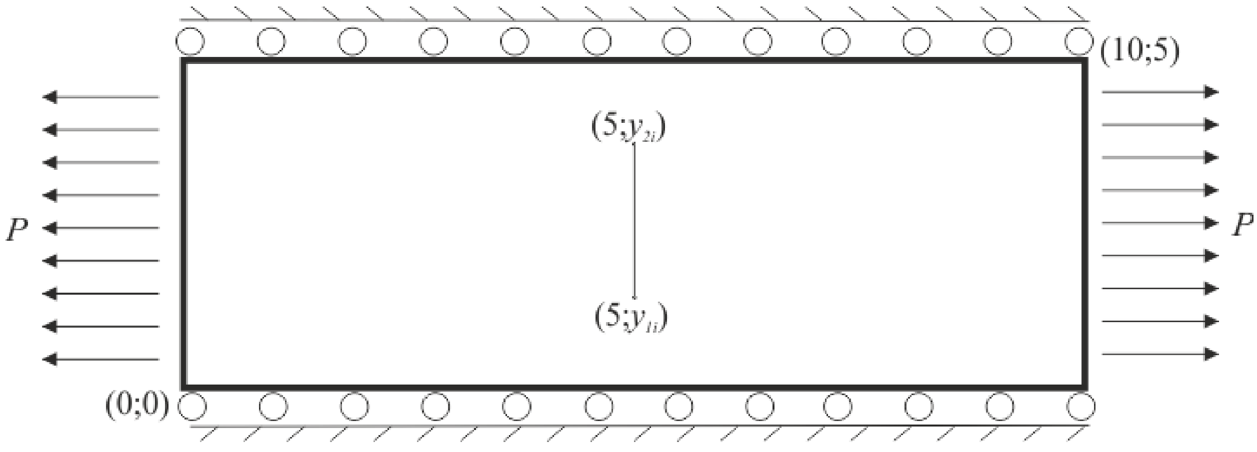
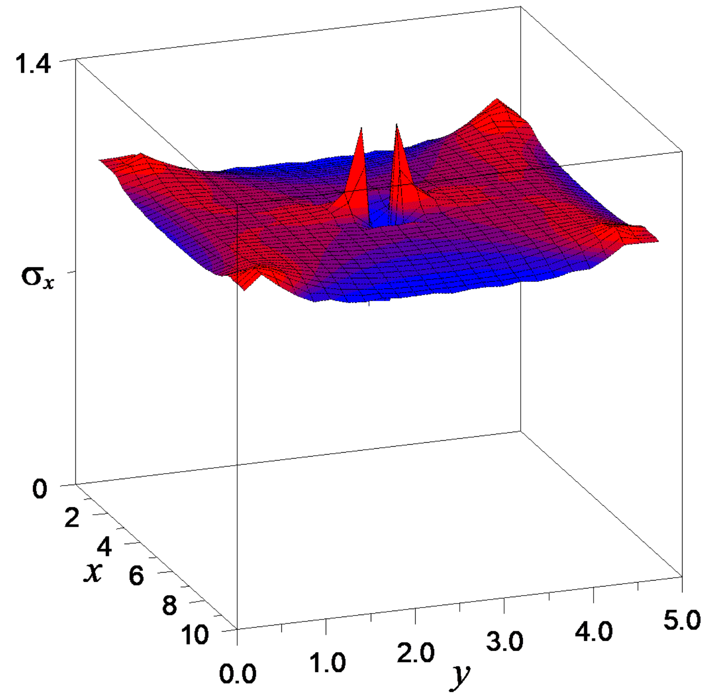
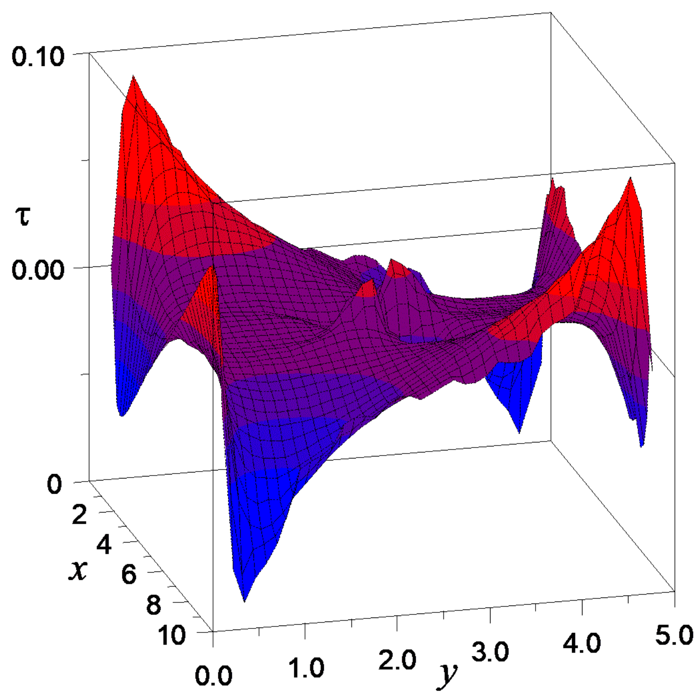


FIGURE 2. Loaded plate with initial crack

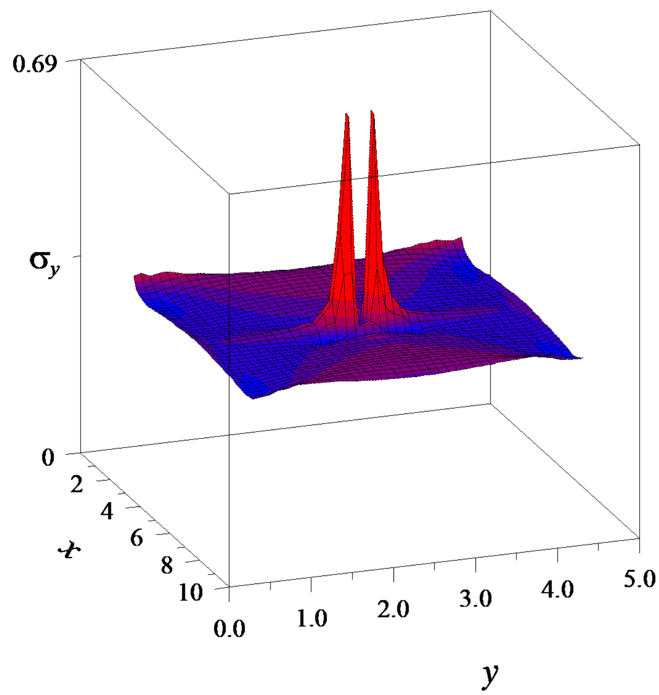
We accept the following values of crack length: l= 0.2; 0.4; 0.6; 0.8; 1; 2.



(a) Stresses along axis x

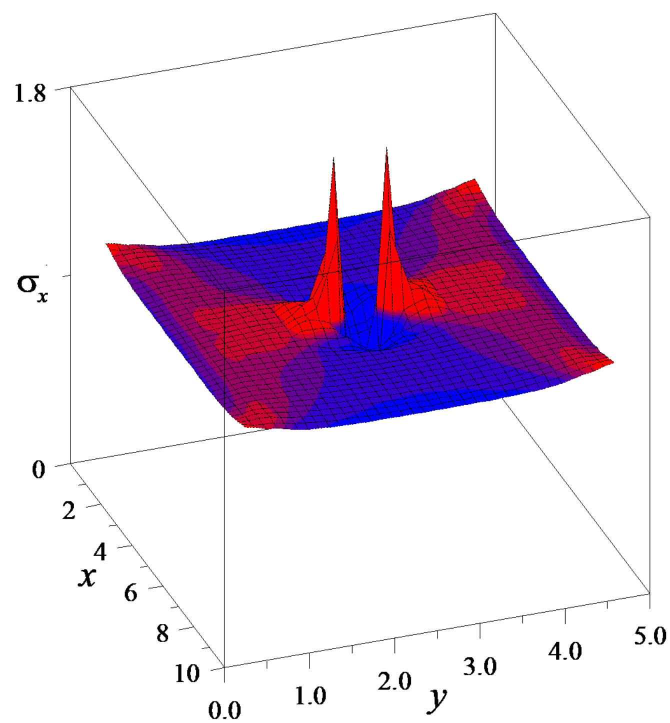


(b) Shear stresses

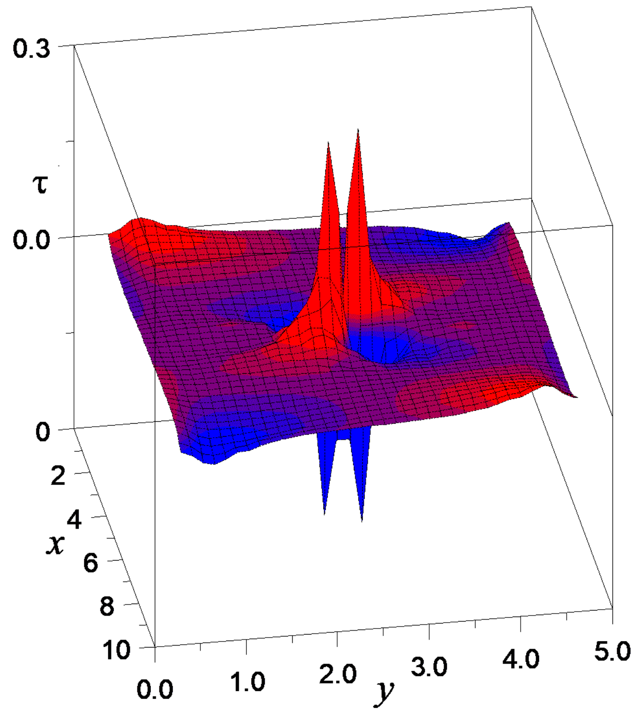


(c) Stresses along axis *y*

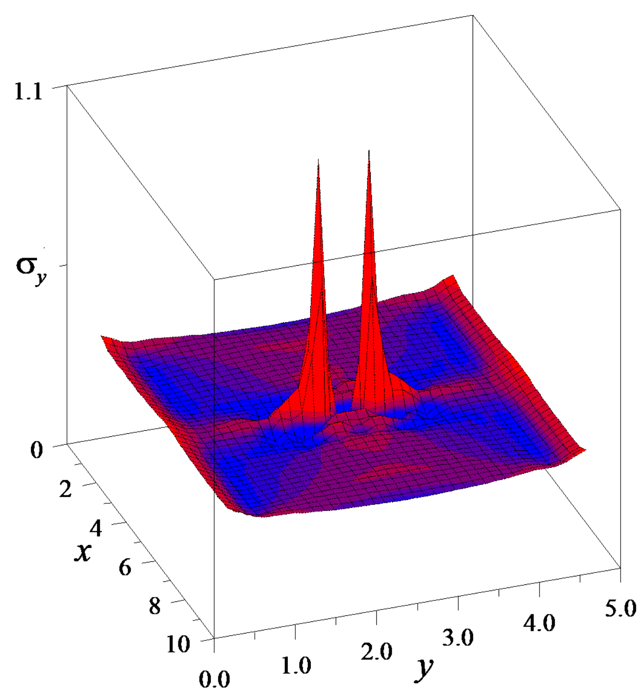
FIGURE 3. Distribution of stresses over the surface of the body ($y_1 = 2.4,\ y_2 = 2.6$)



(a) Stresses along axis *x*

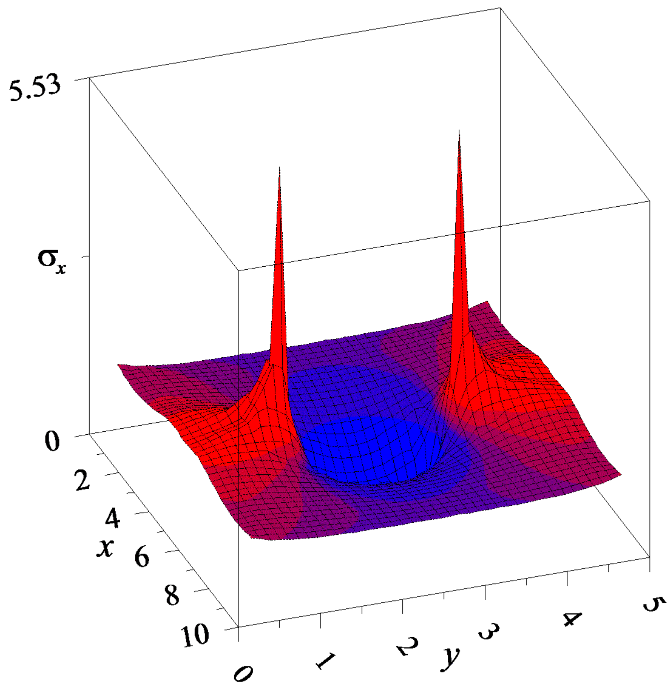


(b) Shear stresses

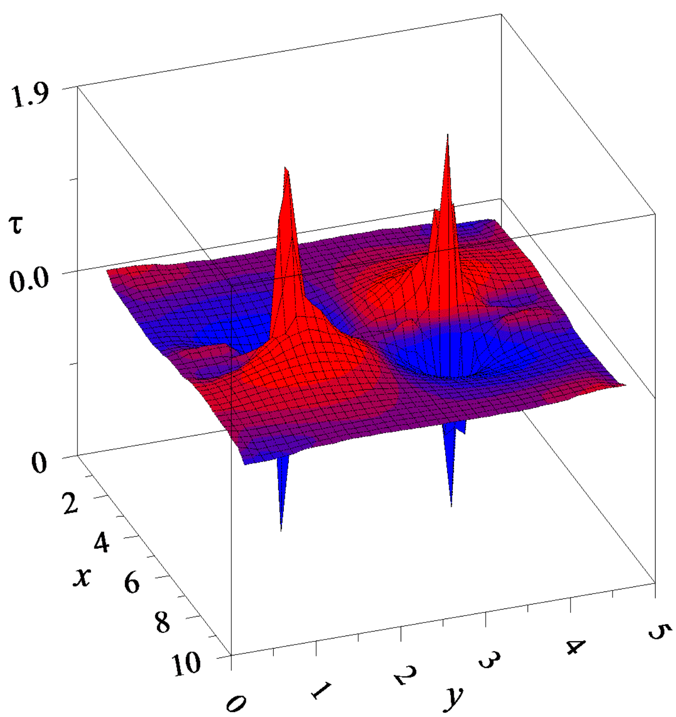


(c) Stresses along axis *y*

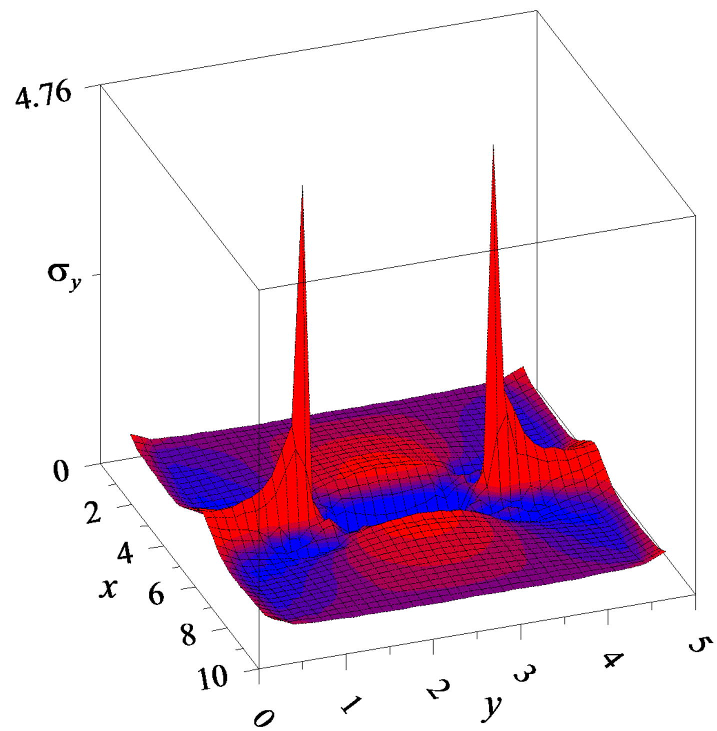
FIGURE 4. Distribution of stresses over the surface of the body (y_1=2.3, y_2=2.7)



(a) Stresses along axis x



(b) Shear stresses



(c) Stresses along axis y

FIGURE 5. Distribution of stresses over the surface of the body (y_1=1.5, y_2=3.5)

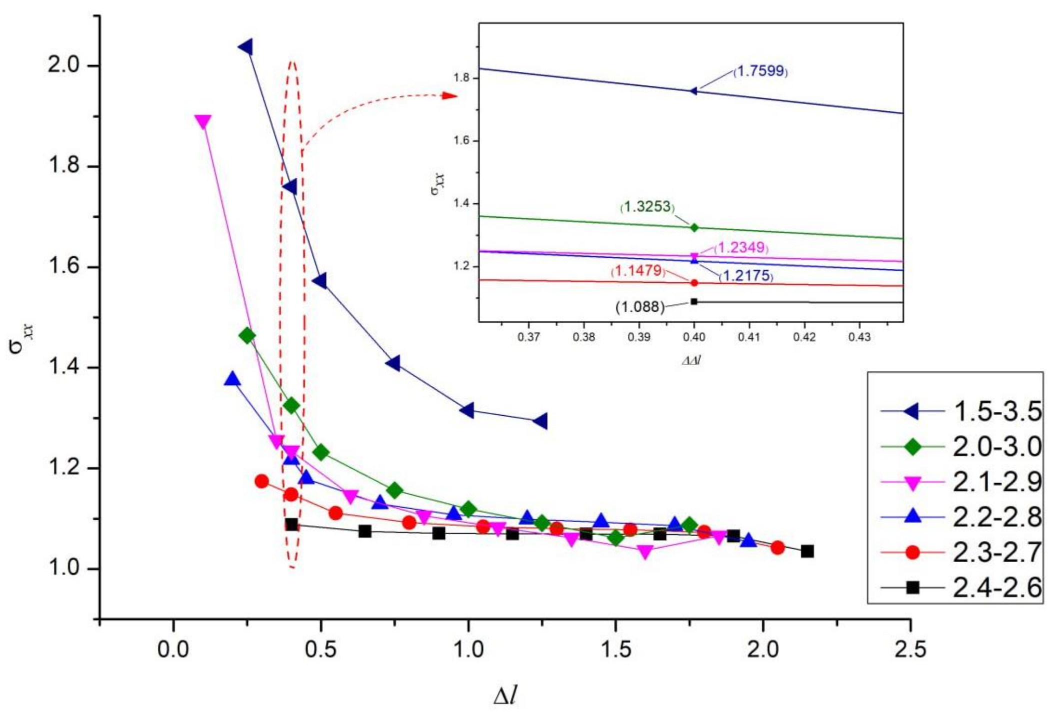
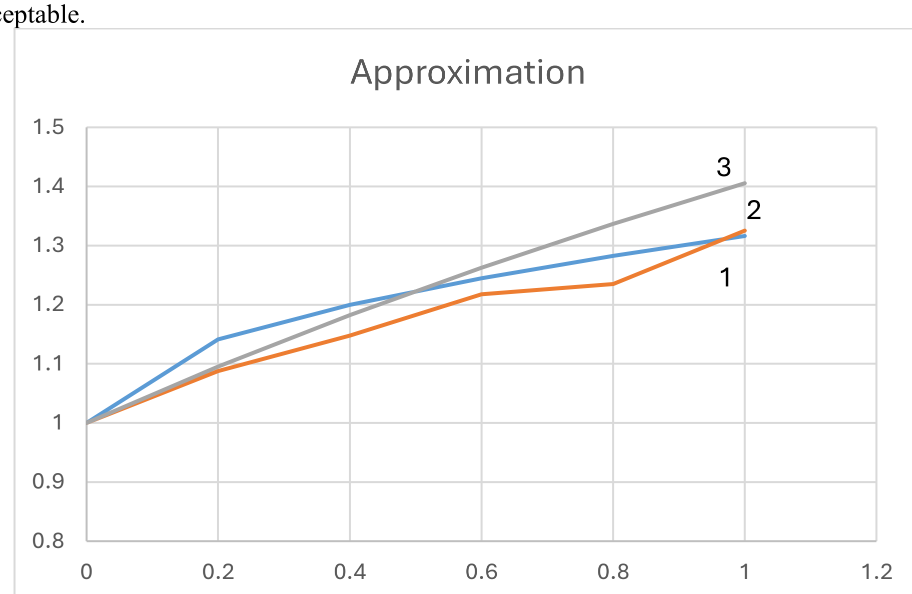


FIGURE 6. Increase in the normal stress values at the crack tip

ANALYSIS OF RESULTS

Figure 6 clearly illustrates that the stress values at a distance of $\Delta l = 1$ from the crack converge, indicating that the size of the crack does not significantly affect the stress distribution in this region (as per the Saint-Venant principle). An exception arises when the crack length is l=2, which is 40% of the plate's characteristic size. In this scenario, the crack emerges as a dominant feature in the material dimensions, strongly influencing the stress-strain distribution throughout the entire body. The crack (linear) can be considered of the limit size when up to 20% of it has a local effect.

The stress values in the immediate vicinity of the crack tip are significantly impacted by the size of the crack, as might be expected. When the crack size doubles, the normal stress value increases from 5.5% to 7.7% at a distance of $\Delta l = 0.4$ from the crack tip. At this distance, calculations based on the theory of brittle fracture (linear problems) are considered acceptable.



1 – numerical calculation, 2 – approximation 1, 3 – approximation 2

FIGURE 7. Normal stresses vs the crack length

The growth of stresses at the crack tip, depending on the crack length, can be approximated as follows:

$\sigma_{yy} = P\left(1 + \ln\left(\frac{l}{H}k + 1\right)\right)$, $k = H/5l$ – logarithmic approximation (No. 3 on the graph);

$\sigma_{yy} = P\left(1 + \sqrt{kl}\right)$, $k = 0.02/(lH)$ – approximation through the root (No. 2 on the graph).

CONCLUSION

The dependence of stress concentration on the crack length and boundary conditions is determined in the study presented.

The boundary conditions do not significantly affect the stress concentration near the singularity. This is due not only to the action of the Saint-Venant principle, but also to the high stress concentration in which the influence of boundary conditions is not dominant.

This is confirmed by experiments for shell structures conducted under cryogenic temperatures, where the factor of the principles of brittle fracture is undeniable.

In general, the paper proposes a dependence of stress concentration on crack length. This approximation can be used as a first approximation in determining possible stress concentrations in plane problems of brittle fracture.

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