Problems of Flexural Vibrations of Plate Structures with Beam Elements

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Abstract. The article is devoted to the formulation and development of a method for solving the problem of vibrations of a plate structure with reinforcing beam elements at the edges, which is under the action of a dynamic impact applied to its base. The main relationships and equations of motion of plates and beam elements are given within the framework of Timoshenko's theory, constructed with respect to forces and moments. As a calculation, bending vibrations of a structure near the resonant state are considered.

INTRODUCTION

The development of dynamic spatial models of buildings and structures whose deformation is of a spatial nature is one of the most complex current problems in mechanics of deformable bodies and dynamics of structures. Various methods have been developed for calculating buildings and structures for seismic effects taking into account important factors.

The article [1] presents a technology for producing anorthite-based building ceramics using semi-dry pressing of powder based on sintering a raw mix consisting of low-melting clay and blast furnace sludge (BFS) in various proportions. The manufactured ceramic samples are sintered at a temperature of 1050 °C. The properties of the raw mix are studied to increase the content of the anorthite phase in the ceramic samples.

The article [2] proposes a new approximate method based on analytical formulas for estimating the ultimate strength of reinforced panels by studying the mechanisms of panel failure. The results of calculations performed by the proposed method and the finite element method are compared. Very good agreement is obtained for all failure scenarios studied.

The article [3] considers the contact interaction of deformable building structures or their parts. The subject of the study is the formulation of the contact interaction problem as a linear complementarity problem. An extension of the existing formulations of the problems of frictionless contact and contact with a known friction boundary in the form of a linear complementarity problem to the formulation of frictional contact is proposed. Ultimately, a heuristic formulation of the contact problem with friction in the form of a linear complementarity problem is obtained.

The article [4] presents finite element equations for a variationally consistent higher-order beam theory for the static and dynamic behavior of rectangular beams. It is shown that full integration of shear stiffness terms leads to the restoration of the Kirchhoff constraint for thin beams without introducing false blocking constraints.

In [5], the dynamic characteristics and oscillations of various axisymmetric and flat structures are considered, taking into account different geometries, spatial factors and inelastic properties of materials.

The article [6] is devoted to the analytical modeling of a simply supported multilayer beam with an expanded-conical cross-section. A simplified analytical model of this beam was developed with the exclusion of the shear effect.

In article [7], based on theoretical and experimental studies, the influence of seismic loads of varying intensity and frequency on wooden frame-type buildings was studied. The seismic resistance of the building was assessed taking into account the properties of wooden vertical posts, lower and upper frames, as well as floor and covering elements.

The paper [8] evaluates classical and improved finite plate elements for bending and vibration of laminated composites and layered structures. About 20 plate finite elements were implemented and compared: classical ones based on displacement assumptions are compared with improved mixed elements, which are formulated based on Reissner's mixed variational theorem.

In [9], the influence of the interaction of foundation plates with the base was studied, and in [10], the stress-strain state of asymmetrically layered plates with controlled forces interacting with a sandy the base was considered.

Article [11] process of free and forced vibrations of a beam system with intermediate supports was studied. Analytical solutions were obtained to ensure effective results in design work.

In references [12, 13] are devoted to the development of a theory and method for calculating thick plates. A theory and a method were developed to assess the stress-strain state of thick plates without simplifying hypotheses within the framework of three-dimensional theory of elasticity. When constructing a theory, all components of strains and stresses arising from the nonlinearity of the law of displacement distribution along the plate thickness were taken into account. The equations of motion of the plate were constructed with respect to forces, moments and bimoments. The solution method was based on exact expressions in trigonometric functions.

The studies in [14-16] are devoted to the numerical solution of the problem of transverse vibrations of a multi-story building within the framework of a continuous plate model of a solid slab under seismic influence. As a dynamic model of the building, a cantilever anisotropic plate is proposed, the theory of which was developed within the framework of the three-dimensional dynamic theory of elasticity and considers not only structural forces and moments but also bi-moments.

STATEMENT OF THE PROBLEM

The problem is posed of bending-shear vibrations of a plate structure with reinforcing beam elements at the edges, which is under the action of a dynamic impact applied to its base in the form of a displacement of the base directed along its normal (fig. 1).

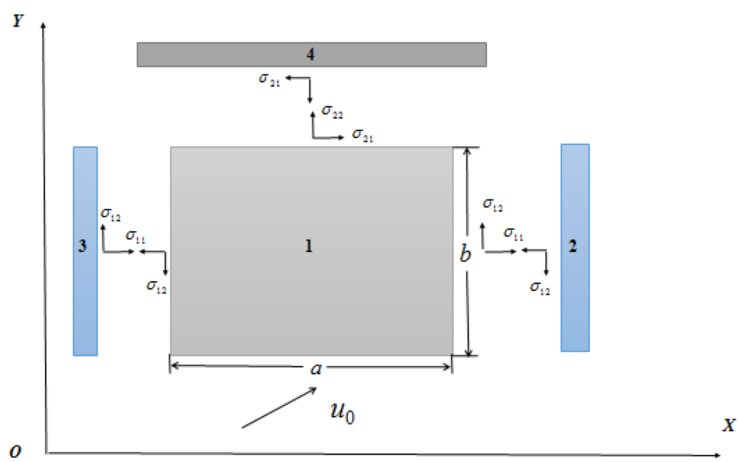

FIGURE 1. Graphs of changes over time t of normal displacement values $\tilde{r}$ at the highest point of a nine-story building near resonant oscillations with a frequency p_1=3.301 Hz.

Let us write down the equations of motion of the plate with respect to bending, torque moments and with respect to shear forces:

$$\frac{\partial M_{11}}{\partial x_1}+\frac{\partial M_{12}}{\partial x_2}-Q_{13}=\frac{H^3}{12}\rho\frac{\partial^2\varphi}{\partial t^2},$$
$$\frac{\partial M_{21}}{\partial x_1}+\frac{\partial M_{22}}{\partial x_2}-Q_{23}=\frac{H^3}{12}\rho\frac{\partial^2\psi}{\partial t^2}. \tag{1}$$

where ρ and $H=2h$ - plate material density.

The equation for the shear forces is written as

$$\frac{\partial Q_{13}}{\partial x_1}+\frac{\partial Q_{23}}{\partial x_2}-q_3=\rho H\frac{\partial^2 w}{\partial t^2} \tag{2}$$

Bending, shear moments M_{11}, M_{22}, M_{12} and shear forces Q_{13}, Q_{23} are defined as follows:
The expressions for bending and shear moments are as follows:

$$M_{11}=\int_{-h}^{h}\sigma_{11}zdz,\quad M_{22}=\int_{-h}^{h}\sigma_{22}zdz,\quad M_{12}=M_{21}=\int_{-h}^{h}\sigma_{12}zdz \tag{3}$$

The expressions for the shear forces are written as:

$$Q_{13}=k^2\int_{-h}^{h}\sigma_{13}dz,\quad Q_{23}=k^2\int_{-h}^{h}\sigma_{23}dz \tag{4}$$

where k^2 - coefficient characterizing the transverse shear of the plate.

By integrating in (3), we obtain expressions for the bending and shear moments for an orthotropic plate

$$M_{11}=D_{11}\left(\frac{\partial\varphi}{\partial x_1}+\nu_{12}\frac{\partial\psi}{\partial x_2}\right),\quad M_{22}=D_{22}\left(\frac{\partial\psi}{\partial x_2}+\nu_{12}\frac{\partial\varphi}{\partial x_1}\right),$$
$$M_{12}=M_{21}=D_{12}\left(\frac{\partial\varphi}{\partial x_2}+\frac{\partial\psi}{\partial x_1}\right). \tag{5}$$

where the cylindrical rigidity of the orthotropic plate has the expressions

$$D_{11}=\frac{E_1H^3}{12(1-\nu_{12}\nu_{21})},\quad D_{22}=\frac{E_2H^3}{12(1-\nu_{12}\nu_{21})},\quad D_{12}=\frac{G_{12}H^3}{12}$$

Similarly, performing integration in (4), we obtain expressions for the shear forces

$$Q_{13}=k^2G_{13}H\left(\varphi+\frac{\partial w}{\partial x_1}\right),\quad Q_{23}=k^2G_{23}H\left(\psi+\frac{\partial w}{\partial x_2}\right) \tag{6}$$

The system of equations of a standing beam 2 with respect to the bending moment, shear force and contact forces is written as:

$$\frac{\partial M_{22}^{(2)}}{\partial x_2}-Q_{12}^{(2)}-h_2\sigma_{12}^{(2)}=-\frac{\rho_2H_2^3}{12}\ddot{\varphi}_2^{(2)},\ \frac{\partial Q_{12}^{(2)}}{\partial x_2}-\sigma_{11}^{(2)}=\rho_2H_2\ddot{W}^{(2)} \tag{7}$$

where $Q_{12}^{(2)}, M_{22}^{(2)}$ - shear force and bending moment of standing beam 2, which are determined through unknown functions of the angle of rotation and deflection of the points of the middle surface: $\varphi_2^{(2)}$, $W^{(2)}$, ρ_2 and $H_2=2h_2$ density and thickness of standing beam 2.

The system of equations of a standing beam 3 of bending moment, shear force and contact forces is written as:

$$\frac{\partial M_{22}^{(3)}}{\partial x_2}-Q_{12}^{(3)}-h_3\sigma_{12}^{(3)}=-\frac{\rho_3H_3^3}{12}\ddot{\varphi}_3^{(3)},\frac{\partial Q_{12}^{(3)}}{\partial x_2}+\sigma_{11}^{(3)}=\rho_3H_3\ddot{W}^{(3)}. \tag{8}$$

where $Q_{12}^{(3)}$, $M_{22}^{(3)}$ - shear force and bending moment of standing beam 3, which are determined through unknown functions $\varphi_2^{(3)}$, $W^{(3)}$, ρ_3 and $H_3=2h_3$ density and thickness of the standing beam 3.

Similarly, a system of equations for the vibrations of the beam 4 (Figure 1) was obtained with respect to the bending moment, shear force and contact forces in the form:

$$\frac{\partial M_{11}^{(4)}}{\partial x_2}-Q_{21}^{(4)}-h_4\sigma_{12}^{(4)}=-\frac{\rho_4H_4^3}{12}\ddot{\varphi}_2^{(4)},\ \frac{\partial Q_{21}^{(4)}}{\partial x_2}-\sigma_{22}^{(4)}=\rho_4H_4\ddot{W}^{(4)}. \tag{9}$$

where $Q_{21}^{(4)}$, $M_{11}^{(4)}$ - shear force and bending moment of the beam, which are determined through unknown functions of the angle of rotation and deflection of the points of the middle surface of the beam $\varphi_1^{(4)}$, $W^{(4)}$; ρ_4 and $H_4=2h_4$ density and thickness of the beam 4.

SOLUTION METHOD

The problems are solved by the finite difference method. To approximate the derivatives of displacements with respect to spatial coordinates, we use the formulas of central difference schemes.

To approximate the first derivatives, the following expressions are used relative to the central points:

$$\frac{\partial f_{i,j}^k}{\partial x_1}=\frac{f_{i+1,j}^k-f_{i-1,j}^k}{2\Delta x_1},\quad \frac{\partial f_{i,j}^k}{\partial x_2}=\frac{f_{i,j+1}^k-f_{i,j-1}^k}{2\Delta x_2} \tag{10}$$

in here $\Delta x_1=\frac{a}{N},\ \Delta x_2=\frac{b}{M}$ – calculation step, $N,\ M$ – number of divisions.

To approximate the derivatives of stresses, forces, moments and bimoments, central finite-difference schemes on half-steps are used, which have the second order of accuracy:

$$\frac{\partial F_{i,j}^k}{\partial x_1}=\frac{F_{i+\frac{1}{2},j}^k-F_{i-\frac{1}{2},j}^k}{\Delta x_1},\quad \frac{\partial F_{i,j}^k}{\partial x_2}=\frac{F_{i,j+\frac{1}{2}}^k-F_{i,j-\frac{1}{2}}^k}{\Delta x_2}\quad \left(i=1,N;\ \ j=1,M\right) \tag{11}$$

in here $\Delta x_1=\frac{a}{N},\ \Delta x_2=\frac{b}{M}$.

The conditions for the equality of the force factors of the plate to zero at the edges free from supports are approximated by the following expressions:

$$F_{N+\frac{1}{2},j}^k+F_{N-\frac{1}{2},j}^k=0\ \left(j=1,M\right);\quad F_{i,M+\frac{1}{2}}^k+F_{i,M-\frac{1}{2}}^k=0\quad \left(i=1,N\right) \tag{12}$$

When using formulas (11) and (12), it is necessary to approximate the derivatives of the generalized displacement functions at the central point between two points x_i and x_{i+1} also y_j and y_{j+1}. In these cases we use formulas (10), replacing accordingly i – to $i-\frac{1}{2}$ and j – to $j-\frac{1}{2}$.

$$\frac{\partial f_{i-\frac{1}{2},j}^k}{\partial x_1}=\frac{f_{i,j}^k-f_{i-1,j}^k}{\Delta x_1},\quad \frac{\partial f_{i,j-\frac{1}{2}}^k}{\partial x_2}=\frac{f_{i,j}^k-f_{i,j-1}^k}{\Delta x_2},\quad \left(i=1,N;\ \ j=1,M\right) \tag{13}$$

$$\frac{\partial f_{i-\frac{1}{2},j}^k}{\partial x_2}=\frac{\partial}{\partial x_2}\left(\frac{f_{i,j}^k+f_{i-1,j}^k}{2}\right),\quad \frac{\partial f_{i,j-\frac{1}{2}}^k}{\partial x_1}=\frac{\partial}{\partial x_1}\left(\frac{f_{i,j}^k+f_{i,j-1}^k}{2}\right),\quad \left(i=1,N;\ \ j=1,M\right) \tag{14}$$

We will represent the second derivative with respect to time, using a finite difference expression, in the form:

$$\frac{\partial^2 f_{i,j}^k}{\partial t^2}=\frac{f_{i,j}^{k+1}-2f_{i,j}^k+f_{i,j}^{k-1}}{\Delta t^2} \tag{15}$$

ANALYSIS OF NUMERICAL RESULTS

We assume that at the base of the plate structure in the horizontal direction along its length the displacement of the base is specified $u_0(t)$ in the form of:

$$u_0(t)=A_0\sin(\omega_0 t). \tag{16}$$

where A_0 and ω_0 – amplitude and frequency of the base displacement. We present the contact conditions between the elements of the structures and the boundary conditions in the base. In the zone of connection of plate elements and post-and-beam elements we have the following contact kinematic conditions:

$$\begin{aligned}
&u_0(t)=A_0\sin(\omega_0 t).u_1(x_1,x_2,t)_{x_1=0}=W^{(3)}(x,t),\ \ u_2(x_1,x_2,t)_{x_1=0}=\frac{h_3}{2}\varphi^{(3)},\\
&u_1(x_1,x_2,t)_{x_1=a}=W^{(2)}(x,t),\ \ u_2(x_1,x_2,t)_{x_1=0}=-\frac{h_2}{2}\varphi^{(2)},\\
&u_2(x_1,x_2,t)_{x_2=b}=W^{(4)}(x,t),\ \ u_2(x_1,x_2,t)_{x_2=b}=-\frac{h_4}{2}\varphi^{(4)}.
\end{aligned} \tag{17}$$

where $W(x,y,t)$ – displacement of bending plate elements. $W^{(i)},\ \varphi^{(i)},\ i=2,3,4.$ – functions of deflections and rotation angles of beam elements.

Boundary conditions on the base of a plate structure $x_2 = 0$ we write it down as for rigid clamping. The lower part of the building moves together with the base

$$u_1(x_1, x_2, t)_{x_2=0} = u_0(t), \quad u_2(x_1, x_2, t) = 0. \tag{18}$$

The equations of bending vibrations of post-and-beam elements (17) - (18) are adopted as contact conditions between transverse and longitudinal plate elements.

The initial conditions of the problem are assumed to be zero. Note that the stress expressions for anisotropic plates are written in the following form:

$$\sigma_{11} = E_{11}\frac{\partial u_1}{\partial x_1} + E_{13}\frac{\partial u_3}{\partial z}, \sigma_{33} = E_{31}\frac{\partial u_1}{\partial x_1} + E_{33}\frac{\partial u_3}{\partial z}, \quad \sigma_{13} = G_{13}\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x_1}\right). \tag{19}$$

We assume that all beam elements are made of reinforced concrete with the modulus of elasticity *E=20000 MPa*, density *ρ=2.5 t/m³* and Poisson's ratio *ν =0.25*. The material of the plate is made of reinforced concrete: modulus of elasticity *E=7500 MPa*, density *ρ=1.2 t/m³*, Poisson's ratio *ν =0.25*. The dimensions of the plate structure: thickness, length and height of the structure, are respectively taken to be equal to *H=0.3 m, a=3* and *4 m; b=3 m*.

For convenience, dimensionless coordinates are introduced using the formulas $x = x_1/a, \quad y = x_2/b, \quad \tau = ct/H$. In the calculations, the calculation step for dimensionless coordinates is taken to be equal to $\Delta x = \Delta y = 1/32, \quad \Delta\tau = k\, c/H \min(\Delta x, \Delta y)$. The stability of the calculation in dimensionless time is ensured by an explicit scheme at a step $\Delta\tau = 0.01$.

Figure 2 shows the graphs of the change in displacement ψ_1 from dimensionless time at equidistant points of plate structures to the resonant state, obtained on the basis of a two-dimensional problem of elasticity theory.

This two-dimensional problem using the bimoment theory of plates along the horizontal coordinate x_2 is reduced to a one-dimensional problem and is solved on the basis of an explicit scheme of the finite difference method.

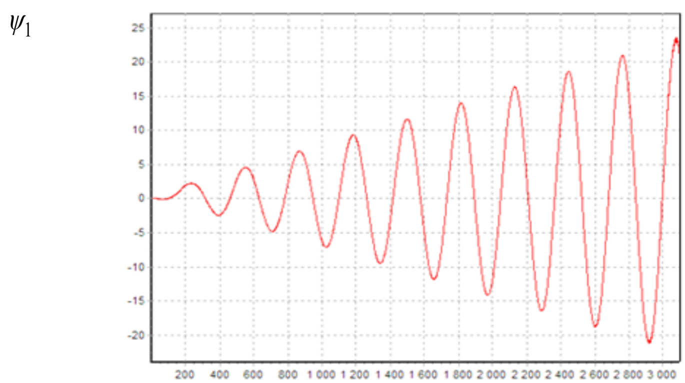

FIGURE 2. Graphs of displacement changes ψ_1 time *t* at equidistant points of the lateral faces of plate structures.

Table 1 shows the first three values of the natural frequency. ω_0 and period T_0 natural vibrations of a plate structure under transverse bending for different values of its small size in plan a.

TABLE 1. The first three values of natural frequency ω_1 and period T_1 natural vibrations of a plate structure under transverse bending for different values of its size in plan a.

$a, \ (m)$	$\omega_1, \ rad/\sec$	$\nu_1 = \omega_0/2\pi, \ 1/\sec,$	$T_1 = 1/\nu_0, \ \sec$
	55.09	8.01	0.123
3	67.81	11.03	0.831
	74.70	12.21	0.715
	45.62	8.89	0.132
4	66.91	10.28	0.091
	69.22	11.62	0.132

The calculations of the plate structure were performed for a seven-point earthquake, for which the acceleration amplitude is equal to $a_0 = k_c g$, where $k_c = 0.1$, g - acceleration of gravity. The amplitude of the base displacement is determined by the formula $A_0 = a_0 / \omega_0^2$.

CONCLUSION

Within the framework of the theory of plates and beams, a dynamic problem of oscillations of a plate structure under the action of a load specified by the displacement of the base according to the harmonic law is set and solved. A numerical method for solving the problem using an explicit scheme of the finite difference method is developed. Based on the application of the resonance method, natural frequencies and oscillation periods of a plate structure with reinforcing beam elements at its edges are found.

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