Analysis of the Stress-Strain State of an Elastoplastic Plate Subjected to Parabolic Loading

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Abstract. The paper considers a numerical solution to the problem of tension of an elastoplastic plate under a parabolic load. The elastoplastic boundary value problems are formulated based on the deformation theory of plasticity and the flow theory. The discrete analog of the boundary value problem is constructed using the finite difference method and solved by an iterative method.

INTRODUCTION

The formulation of boundary value problems in plasticity theory depends on the chosen constitutive relations specifically, on whether the deformation theory or the flow theory is employed. Within the framework of the deformation theory of plasticity, a boundary value problem is analogous to an elastic one. It consists of the equilibrium equations, the constitutive relations of the corresponding theory (replacing Hooke's law), the Cauchy relations, and the boundary conditions. The solution of such a problem can be reduced to a sequence of elastic problems with a variable nonlinear right-hand side. The so-called method of elastic solutions was first proposed by A.A. Ilyushin [1]. This method and its various modifications have been extensively studied in the works of B.E. Pobedry [2], D.L. Bykov [3], V.K. Kabulov [4], S.V. Sheshenin [5], among others.

In the case of the flow theory, the constitutive relations establish a relationship between the increments of stress and strain tensors. Therefore, for a correct formulation of the boundary value problem, the equilibrium equations, Cauchy relations, and boundary conditions must be expressed in terms of the increments of the unknown quantities. It should be noted that the external load is applied in several increments, and the overall solution is constructed as a superposition of the results corresponding to each load increment. The method for solving such problems in terms of increments is commonly known as the method of successive loadings, which has been discussed in detail in [6–9].

The formulation of thermoplastic problems also differs depending on the plasticity theory adopted. In thermoplastic boundary value problems, temperature-dependent terms can be treated as body forces, and the solution obtained by the method of elastic solutions closely resembles that of a conventional elastic problem. Such problems are typically referred to as uncoupled problems of plasticity theory, where the temperature field is assumed to be known as the solution of the heat conduction equation. If the temperature field is unknown, the heat flux equation with the corresponding boundary and initial conditions must be added to the governing equations. In this case, the equilibrium equation is replaced by the equation of motion, resulting in a coupled thermoplastic problem formulated within the deformation theory.

In the case of the flow theory, to formulate a coupled thermoplastic problem, the equations of motion, Cauchy relations, and heat flux equations, as well as the initial and boundary conditions, must be differentiated—i.e., expressed in terms of increments—and considered together with the incremental constitutive relations. Numerical solutions to coupled thermoplastic problems for isotropic and anisotropic materials have been presented in [10–17].

MATERAILS AND METHODS

In general, a plastic boundary value problem consists of the equilibrium equation [2]

$$\sigma_{ij,j} + X_i = 0, \tag{1}$$

a nonlinear constitutive relation representing a tensor function between the stress and strain tensors

$$\sigma_{ij} = f(\varepsilon_{ij}), \tag{2}$$

the Cauchy relations

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{3}$$

and the boundary conditions

$$u_i\big|_{\Sigma_1} = u_i^0, \tag{4}$$

$$\sigma_{i,j}\eta_j\big|_{\Sigma_2} = s_i^0, \tag{5}$$

where, u_i –the displacement components; X_i, S_i – the body and surface forces, respectively; Σ_1, Σ_2 – are parts of the surface Σ of the body V; $n_{i,j}$ – the outward normal to the surface Σ_2 of the body V.

Let us consider the formulation of a boundary value problem according to the deformation theory of plasticity for isotropic materials. Typically, the dependence $\sigma_u = \Phi(\varepsilon_u)$, known as the stress–strain diagram, is determined experimentally from uniaxial tension or torsion tests and characterizes the process of plastic deformation. In numerical implementations of boundary value problems, it is convenient to represent the stress–strain diagram $\sigma_u = \Phi(\varepsilon_u)$ a piecewise-linear form [18]

$$\sigma_u = 2\mu\varepsilon_u + 2(\mu - \mu^{'})(\varepsilon_u - \varepsilon_u^*) \text{ for } \varepsilon_u \geq \varepsilon_u^*. \tag{6}$$

Then the constitutive relations of the deformation theory can be written as

$$\sigma_{ij} = \sigma\delta_{ij} + \frac{\sigma_u}{\varepsilon_u}e_{ij}, \quad \sigma = K\theta, \quad \sigma_u = \Phi(\varepsilon_u) \tag{7}$$

Substituting the piecewise-linear form of we obtain

$$\sigma_{ij} = \lambda\theta\delta_{ij} + 2\mu\varepsilon_{ij} - 2(\mu - \mu^{'})(1 - \frac{\varepsilon_u^*}{\varepsilon_u})e_{ij} \text{ for } \varepsilon_u \geq \varepsilon_u^*. \tag{8}$$

When formulating boundary value problems based on the flow theory of plasticity, the constitutive relation (2) in the system (1) – (5) is replaced by a relation between the differentials of the stress and strain tensors. Accordingly, to formulate the boundary value problem, the equilibrium equations, Cauchy relations, and boundary conditions must be differentiated.

As an example, let us consider the boundary value problem of the flow theory of plasticity with a yield surface in the strain space. Then, the boundary value problem, based on the flow theory, consists of the differential form of the equilibrium equations [18]

$$d\sigma_{ij,j} + dX_i = 0, \tag{9}$$

the constitutive relation

$$d\sigma_{ij} = \lambda d\theta\delta_{ij} + 2\mu d\varepsilon_{ij} - \frac{\mu - \mu^{'}}{\varepsilon_u^2}(e_{kl}de_{kl})e_{ij} \text{ for } F = 0 \text{ and } e_{kl}de_{kl} > 0, \tag{10}$$

the Cauchy relations

$$d\varepsilon_{ij} = \frac{1}{2}(du_{i,j} + du_{i,j}), \tag{11}$$

and the boundary conditions

$$du_i\big|_{\Sigma_1} = du_i^0, d\sigma_{ij}n_j\big|_{\Sigma_2} = dS_i^0. \tag{12}$$

Here, e_{ij} – the deviator of the strain tensor, X_i, S_i^0 – the components of the body and surface forces, *F*– denotes the yield surface, μ – the elastic modulus, μ' – the tangent modulus.

In this section, the well-known Timoshenko–Goodier problem [19] on the tension of a rectangular plate under a parabolic load is generalized for an elastoplastic plate. The boundary value problems are formulated based on Ilyushin's deformation theory [20] and the flow theory of plasticity with a yield surface in strain space [18].

The elastoplastic boundary value problem of Ilyushin's theory of small elastoplastic deformations (Ilyushin's deformation theory) consists of the equilibrium equations, the constitutive relations of the deformation theory, and the Cauchy relations with the corresponding boundary conditions.

PROBLEM FORMULATION

In the case of plane strain, the plastic problem (9) – (12) can be written in terms of displacements as follows

$$\begin{aligned}&(\lambda+2\mu)\frac{\partial^2 u}{\partial x^2}+(\lambda+\mu)\frac{\partial^2 v}{\partial x\partial y}+\mu\frac{\partial^2 u}{\partial y^2}+P_1=0,\\&(\lambda+2\mu)\frac{\partial^2 v}{\partial y^2}+(\lambda+\mu)\frac{\partial^2 u}{\partial x\partial y}+\mu\frac{\partial^2 v}{\partial x^2}+P_2=0\end{aligned}\tag{13}$$

with the corresponding boundary conditions

$$u\big|_{\Sigma_1}=u^0, v\big|_{\Sigma_1}=v^0,\tag{14}$$

$$\begin{aligned}&\left[\left((\lambda+2\mu)\frac{du}{dx}+\lambda\frac{dv}{dy}-\sigma_{11}^p\right)n_1+\left(\mu(\frac{du}{dy}+\frac{dv}{dx})-\sigma_{11}^p\right)n_2\right]_{\Sigma_1}=S_1.\\&\left[\left(\mu(\frac{du}{dy}+\frac{dv}{dx})-\sigma_{21}^p\right)n_1+\left((\lambda+2\mu)\frac{dv}{dy}+\lambda\frac{du}{dx}-\sigma_{22}^p\right)n_2\right]_{\Sigma_2}=S_2,\end{aligned}\tag{15}$$

where P_i and σ_{ij}^p represent the nonlinear parts of the Lame equations and the constitutive relation (8), respectively, and are given by

$$\begin{cases}\sigma_{ij}^p=2(\mu-\mu^{'})\left(1-\dfrac{\varepsilon_u^*}{\varepsilon_u}\right)e_{ij}\\ P_i=\Sigma_{j=1}^2\dfrac{\partial\sigma_{ij}^p}{\partial x_j}=2(\mu-\mu^{'})\Sigma_{j=1}^2\dfrac{\partial e_{ij}}{\partial x_j}\end{cases}\quad\text{for }\varepsilon_u\geq\varepsilon_u^*.\tag{16}$$

Initially, the boundary value problem (13)–(15) is considered under the condition $\varepsilon_u \geq \varepsilon_u^*$, i.e., within the elastic region. Therefore, the nonlinear quantities associated with plastic deformations are assumed to be zero, meaning that $\sigma_{ij}^p=0$ and $P_i=0.$

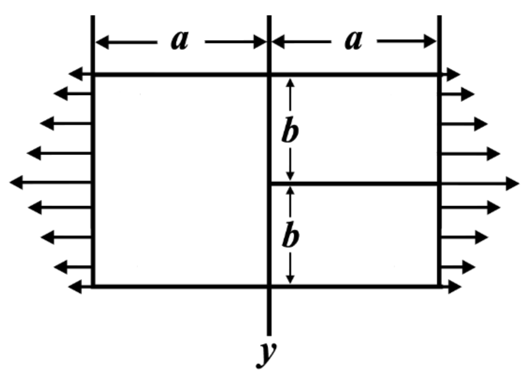

FIGURE 1. Tension of a plastic plate of size $2a\times 2b$ under a parabolic load.

Let a rectangular plate of size $2a\times 2b$ be subjected to a parabolic load (Fig. 1) applied to its opposite edges, while the remaining edges are free of external forces. According to Timoshenko and Goodier [19], the boundary conditions (15) can be expressed as for $x = \pm a$: $$\sigma_{11} = S(1 - \frac{y^2}{a^2}), \sigma_{12} = 0. \tag{17}$$

for $y = \pm b$: $$\sigma_{22} = 0, \sigma_{21} = 0. \tag{18}$$

The boundary conditions (17) – (18) can be expressed in terms of displacements by means of Hooke's law

for $x = \pm a$: $$\begin{cases} \sigma_{11} = (\lambda + 2\mu)\frac{du}{dx} + \lambda \frac{dv}{dy} = S(1 - \frac{y^2}{a^2}), \\ \sigma_{12} = \mu(\frac{du}{dy} + \frac{dv}{dx}) = 0. \end{cases} \tag{19}$$

for $y = \pm b$: $$\begin{cases} \sigma_{22} = (\lambda + 2\mu)\frac{dv}{dy} + \lambda \frac{du}{dx} = 0, \\ \sigma_{21} = \mu(\frac{du}{dy} + \frac{dv}{dx}) = 0. \end{cases} \tag{20}$$

The finite-difference analog of the boundary value problem (19)–(20), resolved with respect to the nodal displacements $u_{i,j}$ and $v_{i,j}$ and incorporating the iterative parameter (k), takes the form

$$u_{ij}^{(k+1)} = \frac{(\lambda + 2\mu)\frac{u_{i+1,j}^{(k)} - u_{i-1,j}^{(k)}}{h_1^2} + (\lambda + \mu)\frac{v_{i+1,j+1}^{(k)} - v_{i-1,j+1}^{(k)} - v_{i+1,j-1}^{(k)} + v_{i-1,j-1}^{(k)}}{4h_1h_2} + \mu\frac{u_{i,j+1}^{(k)} - u_{i,j-1}^{(k)}}{h_2^2} - X_1^{*(k)}}{\frac{2(\lambda + 2\mu)}{h_1^2} + \frac{2\mu}{h_2^2}} \tag{21}$$

$$v_{ij}^{(k+1)} = \frac{(\lambda + 2\mu)\frac{u_{i+1,j}^{(k)} - u_{i-1,j}^{(k)}}{h_1^2} + (\lambda + \mu)\frac{v_{i+1,j+1}^{(k)} - v_{i-1,j+1}^{(k)} - v_{i+1,j-1}^{(k)} + v_{i-1,j-1}^{(k)}}{4h_1h_2} + \mu\frac{u_{i,j+1}^{(k)} - u_{i,j-1}^{(k)}}{h_2^2} - X_1^{*(k)}}{\frac{2(\lambda + 2\mu)}{h_1^2} + \frac{2\mu}{h_2^2}} \tag{22}$$

The boundary conditions for the nodal points can be expressed as

$$\begin{cases} (\lambda + 2\mu)\frac{u_{N_1,j} - u_{N_1-1,j}}{h_1} + \lambda\frac{v_{N_1,j+1} - v_{N_1,j-1}}{2h_2} = S(1 - \frac{y_j^2}{a^2}), \\ \frac{u_{N_1,j+1} - u_{N_1,j-1}}{2h_2} + \frac{v_{N_1,j} - v_{N_1-1,j}}{h_1} = 0. \end{cases} \tag{23}$$

$$\begin{cases} (\lambda + 2\mu)\frac{u_{1,j} - u_{0,j}}{h_1} + \lambda\frac{v_{0,j+1} - v_{0,j-1}}{2h_2} = S(1 - \frac{y_j^2}{a^2}), \\ \frac{u_{0,j+1} - u_{0,j-1}}{2h_2} + \frac{v_{1,j} - v_{0,j}}{h_1} = 0. \end{cases} \tag{24}$$

Finite-difference equations for the remaining edges of the rectangle can be obtained in a similar manner. By solving the boundary conditions with respect to $u_{N_1,j}, u_{0,j}, v_{N_1,j}, v_{0,j}$ and combining them with eqs. (21) – (22), the system can be solved iteratively.

The calculated values of the stresses σ_{11}, obtained from the numerical displacements, are presented in Table 1. The input parameters used in the computations were as follows

$$\lambda = 1.5, \mu = 0.75, a = 1, b = 1, h_1 = 0.2, h_2 = 0.2, N_1 = N_2 = 10. \tag{25}$$

Let us now consider the elastic–plastic problem (13) – (15) within the plastic zone, i.e., taking into account relations (16) under the condition $\varepsilon_u \geq \varepsilon_u^*$. The finite-difference equations corresponding to the elastic–plastic problem are formulated in a similar manner and can be solved using the method of elastic solutions combined with an iterative procedure.

TABLE 1. Comparison of the stress values σ_{11}/S at $x=0$.

Methods	y=0	y=0.2	y=0.4	y=0.6	y=0.8
Iterative method	0.3202	0.4423	0.5899	0.7235	0.8054
Timoshenko–Goodier [9]	0.3404	0.5166	0.6536	0.7515	0.8102

The elastic–plastic problem of stretching a plate under a parabolic load was analyzed for the following dimensionless input parameters

$$E=2, \nu=1/3, \lambda=0.8, \ \mu=0.5, \ \mu^{'}=0.4, \varepsilon_u^*=0.22. \tag{26}$$

The same problem was also formulated on the basis of the flow theory of plasticity with the loading surface in the strain space, using Eqs. (9) – (12). For the two-dimensional case, the corresponding finite-difference equations were derived analogously to the previous (elastic) problem and solved using the method of successive loadings.

Table 2 presents a comparison of the stress values σ_{11}, obtained numerically from the plasticity problems based on the deformation theory and the flow theory of plasticity.

TABLE 2. Comparison of the stress values

Theories	y=0	y=0.2	y=0.4	y=0.6	y=0.8	y=1
Deformation Theory	0.7563	0.7309	0.6590	0.5521	0.4211	0.3050
Flow Theory	0.8035	0.7737	0.6889	0.5796	0.4446	0.3166

According to Tables 3 and 4, the values of the normal stresses σ_{11} at the mid-plane of the plate, corresponding to the deformation theory and the flow theory of plasticity, are compared.

TABLE 3. Values of the stresses $\sigma_{11}(x,y)$ according to the deformation theory of plasticity

	x=-a	x=-4a/5	x=-3a/5	x=-2a/5	x=-a/5	x=0
y=b	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y=4b/5	0.3600	0.3391	0.3656	0.3978	0.4256	0.4352
y=3b/5	0.6400	0.5650	0.5739	0.5784	0.5823	0.5837
y=2b/5	0.8400	0.7339	0.7267	0.7065	0.6952	0.6915
y=b/5	0.9600	0.8359	0.8097	0.7832	0.7665	0.7608
y=0	1.0000	0.8704	0.8377	0.8095	0.7913	0.7850
y=-b/5	0.9600	0.8359	0.8097	0.7832	0.7665	0.7608
y=-2b/5	0.8400	0.7339	0.7267	0.7065	0.6952	0.6915
y=-3b/5	0.6400	0.5650	0.5739	0.5784	0.5823	0.5837
y=-4b/5	0.3600	0.3391	0.3656	0.3978	0.4256	0.4352

TABLE 4. Values of the stresses $\sigma_{11}(x,y)$ according to the flow theory of plasticity

	x=0	x=a/5	x=2a/5	x=3a/5	x=4a/5	x=a
y=b	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
y=4b/5	0.4352	0.4256	0.3978	0.3656	0.3391	0.3600
y=3b/5	0.5837	0.5823	0.5784	0.5739	0.5650	0.6400
y=2b/5	0.6915	0.6952	0.7065	0.7267	0.7339	0.8400
y=b/5	0.7608	0.7665	0.7832	0.8097	0.8359	0.9600
y=0	0.7850	0.7913	0.8095	0.8377	0.8704	1.0000
y=-b/5	0.7608	0.7665	0.7832	0.8097	0.8359	0.9600
y=-2b/5	0.6915	0.6952	0.7065	0.7267	0.7339	0.8400
y=-3b/5	0.5837	0.5823	0.5784	0.5739	0.5650	0.6400
y=-4b/5	0.4352	0.4256	0.3978	0.3656	0.3391	0.3600
y=-b	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

CONCLUSION

Here we provide some basic advice for formatting your mathematics, but we do not attempt to define detailed styles or specifications for mathematical typesetting. You should use the standard styles, symbols, and conventions for the field/discipline you are writing about. Plane plastic problems of tension of a rectangular plate under a parabolic load were formulated within the framework of the deformation theory of plasticity and the flow theory of plasticity with a loading surface in the strain space. Using the finite difference method, the corresponding discrete equations were derived. These finite-difference equations were solved by means of an iterative method in combination with the method of elastic solutions and the method of successive loading.

A comparison of the plastic zones, displacements, and stress values obtained from the deformation theory and the flow theory confirms the validity of the formulated plastic boundary value problems and the reliability of the obtained numerical results.

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