Qualitative research of solutions of free and forced oscillations of hereditarily deformable systems

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Abstract. The paper considers the problems of free and forced oscillations of hereditarily deformable systems with one degree of freedom. The solutions of the dynamic problem of free and forced oscillations are qualitatively investigated.

INTRODUCTION

The problem of oscillations of hereditarily deformable systems with a finite degree of freedom by the method of orthogonal transformations can be reduced to a number of problems about oscillations with one degree of freedom. To do this, we qualitatively investigate solutions of free oscillations of hereditarily deformable systems with one degree of freedom.

The solution of free oscillations of hereditarily deformable systems with one degree of freedom under arbitrary initial conditions has the form:

$$z(t) = c_1 y_1(t) + c_2 y_2(t) \tag{1}$$

where

$$y_1(t) = sin\,\omega\,t + s\varphi(t); y_2(t) = cos\,\omega\,t + c\varphi(t). \tag{2}$$

Here, both $s\varphi(t)$ and $c\varphi(t)$ the sine and cosine functions are of fractional order [1-5]. Using functions (2), new mechanical effects of the general formulation of the problem can be described. It is clear from (1) that if functions $\sin\omega t$ and $\cos\omega t$ describe vibrations of elastic systems, then from (1), (2) it can be seen that functions $s\varphi(t)$ and $c\varphi(t)$ describe the law of internal friction of the system, and from (2) it follows that these functions, together with functions, $s\varphi(t)$ and $c\varphi(t)$ describe free vibrations of viscoelastic systems. In particular, under the initial conditions $z(0) = T_0 = c_2, \dot{z}(0) = 0$ of (2), we have

$$z(t) = T_0\,cos\,\omega\,t + T_0 c\varphi(t). \tag{3}$$

It can be seen from the solution that, according to law (3), free oscillations of viscoelastic systems occur near the curve expressed by the second term of solution (3), i.e. $T_0 c\varphi(t)$. Thus, the following asymptotic representations of the solution for an arbitrary relaxation core show that the function $c\varphi(t)$ over time leads to an asymptotic tendency of the solution to zero [1] and allows us to clearly describe the complete qualitative picture of the phenomenon under study and the law of variation of internal friction of systems over time during oscillatory processes of viscoelastic systems [6-12].

MAIN PART AND RESULTS

Let's start with qualitative studies of solution (3), i.e., the solution related to free oscillations of viscoelastic systems. If the nature of the free oscillations of the system is known, then it is possible to judge its inherent internal properties that manifest themselves under the influence of external disturbances [1]. To do this, it is necessary to study the properties of the function $c\varphi(t)$, knowing which, it is not difficult to study the function $s\varphi(t)$, since [13]:

$$y_1(t) = \sin\omega t + s\varphi(t) = \omega\int_0^t [\cos\omega t + c\varphi(t)]dt = \omega\int_0^\tau y_2(\tau), \tag{4}$$

where

$$y_2(t) = \cos\omega t + c\varphi(t),$$

$$c\varphi(t) = \sum_{k=1}^{\infty}(-1)^k \frac{(\omega\sqrt[2k]{c_k(t)})^{2k}}{(2k)!}; c_k(t) = \int_0^1 (1-s)^{2k} L_k^{`}(ts)ds; \tag{5}$$

$$L^{'}(ks) = \sum_{i=1}^{k}(-1)^i c_i^k \phi_i^{`}(ts); \phi_i^{'}(ts) = \int_0^{ts} R(ts-\tau)\phi_i^{`}(t\tau)d\tau.$$

First of all, it can be seen from (5) that $c\varphi(0) = 0$, since from the expression for $C_k(t)$, having integrated in parts, we have

$$C_k(t) = -1 + 2k\int_0^1 (1-S)^{2k-1} L_k(ts)ds = -1 + P_k(t); \tag{6}$$

$$L_k(t) = L_o - \sum_{i=o}^{k-1}\int_0^t R(t-\tau)L_i(\tau)a\tau, L_o = 1. \tag{7}$$

According to (6), the function $c\varphi(t)$ takes the form

$$c\varphi(t) = -\cos\omega t + \sum_{k=o}^{\infty}(-1)^k \frac{(\omega t)!}{2k!} P_k(t).$$

Moreover, the function $P_k(t)$ satisfies the following conditions [14,15]:

1. $P_k(0) = 1$;
2. $0 < P_k(t) < 1$ for any finite k and t;
3. $\lim_{k\to\infty} P_k(t) = 0$ under any circumstances 0
4. $\lim_{t\to\infty} P_k(t) \le \alpha_0^k, 0 < \alpha_0 < 1$

As is known [3,6], the relaxation $\vec{R}(t) = 1 - \int_0^t R(\tau)d\tau$ function has the following properties:

1) the function $\vec{R}(t)$ is defined for all $t \ge 0$,
2) $\vec{R}(t)$ it is not negative when $t \ge 0$,
3) $\vec{R}(t)$ a monotonically decreasing function tending to $t \to \infty$ a certain limit α_0, and $0 < \alpha_0 < 1$.

Using these properties of the function $\vec{R}(t)$ and the recurrent formulas (7), it is not difficult to obtain

$$L_k(0) = 1; L_k(t) \ge 0; L_k(t) \le \left[\vec{R}(t)\right]^k; \lim_{t\to\infty} L_k(t) = \alpha_0^k < \alpha. \tag{8}$$

The first two conditions (8) are obvious; let us show the validity of the last two conditions. Bearing in mind the monotony of the po function $R(t-\tau)$ by $t-\tau$ and continuity $L_k(t)$, from the recurrence relations (7) we obtain

$$L_k(t) = L_{k-1}(t) - L_{k-1}(\xi)\int_o^t R(\tau)d\tau, 0 < \xi \le \tau.$$

or

$$L_k(t) = L_{k-1}(t)[1 - \beta_{k-1}(t,\xi)]\int_o^t R(\tau)d\tau,$$

where $\beta_{k-1}(t,\xi) = L_{k-1}(\xi)/L_{k-1}(t) \ge 1$. Therefore,

$$L_k(t) \le \left[\vec{R}(t)\right]^k, k = 1,2\ldots \tag{10}$$

Moving on to the limit at $t\to\infty$ and using the third property of the function $\vec{R}(t)$, we will verify the validity of all conditions (8).

It remains to prove the validity of condition (9) for the function

$$P_k(t) = 2k\int_0^1 (t-s)^{2k-1} L_k(t-s)ds.$$

It follows from (10) that

$$P_k(t) \le 2k\int_0^1 (t-s)^{2k-1}[\bar{R}(t-s)]^k ds \le [\vec{R}(t)]^k.$$

If we use the properties of the function $\bar{R}(t)$, then by making the limit transition at $k \to \infty$ and $t \to \infty$, we obtain all the conditions (8).

Thus, the proposed approach makes it possible to construct an exact analytical solution of the integro-differential equations of dynamic problems of the linear theory of hereditarily deformable systems in all existing weakly singular kernels of heredity in the form of a convergent fractional power series. It should be noted that solutions (3), together with conditions (8) and (9), contain the necessary information to understand the qualitative nature of time changes in elastic and viscoelastic solutions and the laws of internal friction of dynamic problems of hereditarily deformable systems [17].

In fact, the first terms of the solution (3) contain well-studied information only about elastic solutions, the second term is the law of internal friction of a material and viscoelastic solutions to problems. It follows from the presented function in the form of (5) and the validity of condition (9) that the function $c\varphi(t)$, describing the law of change of internal friction during free oscillations of viscoelastic systems oscillates over time with a slowly increasing amplitude located opposite the phase of the function $\cos\omega t$ and a slight shift in the oscillation frequency, i.e.

$$\lim_{t\to\infty}\omega_1 = \omega, \lim_{t\to\infty} c\varphi(t) = -\cos\omega t. \tag{11}$$

Qualitative patterns are observed for the law of time variation of the function $y_2(t)$.

Calculations based on the addition theorem $y_2(t_n + \Delta t)$ and formulas

$$A_0 = \frac{\Delta t}{2}; A_j = \Delta t; j = \overline{1, n-1}; B_0 = \frac{(\Delta t)^\alpha}{2}; B_k = \frac{(\Delta t)^\alpha}{2}[(k+1)^\alpha - (k-1)];$$

where, $k = \overline{1, j-1}$; $B_j = \frac{(\Delta t)^\alpha}{2}[j^\alpha - (j-1)^\alpha]; k = j$, they confirm the validity of the high-quality paintings shown in the drawing.

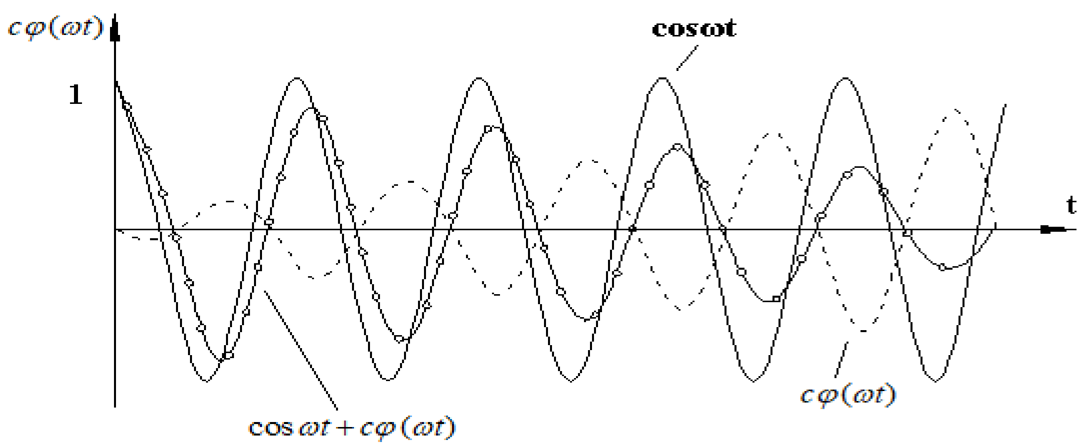

FIGURE 1. The law of time variation of the function $y_2(t)$.

Similar qualitative pictures can be obtained for the function $y_1(t) = \sin\omega t + s\varphi(t)$, and conditions (8) and (9) will be valid for the function $s\varphi(t)$, since [1, 17, 18]

$$y_2(t) = \omega\int_0^t y_1(\tau)d\tau. \tag{14}$$

The limit transition (11) for this case takes the form:

$$\lim_{t\to\infty}\omega_1 = \omega, \lim_{t\to\infty} s\varphi(t) = -\sin\omega t. \tag{15}$$

The law of time variation of a function $y_1(t)$ is characterized by the following qualitative patterns. And in this case, calculations based on the addition theorem $y_1(t_n + \Delta t)$ and formulas

$$y_{1n} = t_n - \omega^2 \sum_{j=0}^{n-1} A_j(t_n - t_j)\left[y_{1j} - \frac{\varepsilon}{\alpha}\sum_{k=0}^{j} B_k\, e^{-\beta t_j} y_{2j-k}\right]$$

where

$$A_0 = \frac{\Delta t}{2}; A_j = \Delta t; j = \overline{1, n-1}; B_0 = \frac{(\Delta t)^\alpha}{2}; B_k = \frac{(\Delta t)^\alpha}{2}[(k+1)^\alpha - (k-1)];$$
$$k = \overline{1, j-1}; B_j = \frac{(\Delta t)^\alpha}{2}[j^\alpha - (j-1)^\alpha]; k = j,$$

the validity of high-quality paintings has been fully confirmed.

Thus, taking into account the inherently deformable properties of the structural material lead to attenuation of free vibrations, while the attenuation rate significantly depends on the singularity parameter [19].

The smaller the singularity parameter of the structural material, the higher the rate of attenuation of natural vibrations.

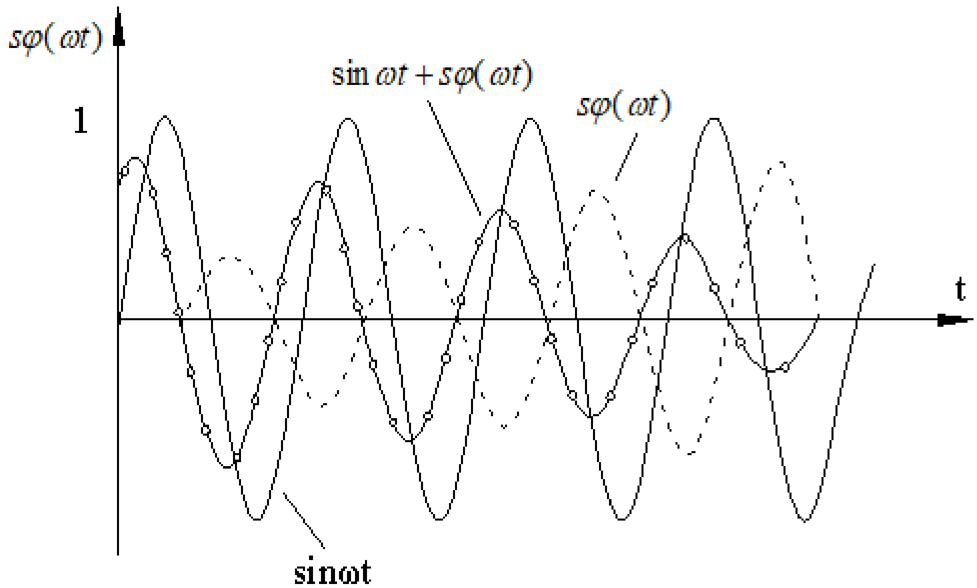

FIGURE 2. The law of time variation of the function $y_1(t)$

We turn to qualitative research on the solution of forced oscillations of hereditarily deformable systems with one degree of freedom.

As is known, the general integral of an inhomogeneous integro-differential equation (IDE) is the sum of the general solution of the corresponding homogeneous IDE and the particular solution of the inhomogeneous IDE under consideration [20]:

$$z(t) = z_{од}(t) + z_{неод}(t), \tag{16}$$

where

$$z_{од}(t) = c_1 y_1(t) + c_2 y_2(t), \tag{17}$$

$$z_{неод} = \frac{1}{\omega}\int_o^t F(\tau) y_1(t-\tau) d\tau. \tag{18}$$

From the above studies, it is clear that the solution of a homogeneous IDE in the presence of internal friction is rapidly decaying, therefore, only a partial solution of z_n, which determines the forced fluctuations of the problem, is of practical interest.

Let us consider a solution in the case of forced oscillations under trivial initial conditions $c_1 = c_2 = 0$. Then from (16) and (18), after some simple transformations, we obtain [1,6]:

$$z(t) = \frac{1}{\omega^2}\left[\frac{d}{d\tau}\int_0^\tau P(t-\tau)F(\tau)d\tau - F(0)\frac{d}{d\tau}\int_0^t P(t-\tau)y_2(\tau)d\tau - \int_0^t \dot{F}(t-\tau)\frac{d}{d\tau}\int_0^\tau P(\tau-s)y_2(s)ds\right]d\tau, \tag{19}$$

where $P(t) = 1 + \int_0^t \Gamma(\tau)d\tau$ - creep function.

In expression (19), the first term is the displacement caused by the force F(t) under its static action, the second is free fluctuations from the starting point as a result of a sudden applied force, and the third term is a dynamic correction caused by changes in the acting force overtime [21].

The exact solution of (19) makes it possible to detect a number of new mechanical effects that cannot be determined by any numerical or approximate analytical methods.

Let's assume that the system is exposed to an external intensity $F(t) = q_0$ −const. Then solution (19) takes the form

$$\bar{z}(t) = P(t) - \frac{d}{dt}\int_0^t P(t-\tau) y_2(\tau) d\tau, \bar{z}(t) = \frac{\omega^2 z(t)}{q_0}. \quad (20)$$

It follows from (3) that the second term in (20) describes a symmetrical decaying oscillatory process. Therefore, it is easy to see from (20) that vibrations of viscoelastic systems under the influence of constant external load pass near the curve of the creep function and attenuate over time along this curve [22].

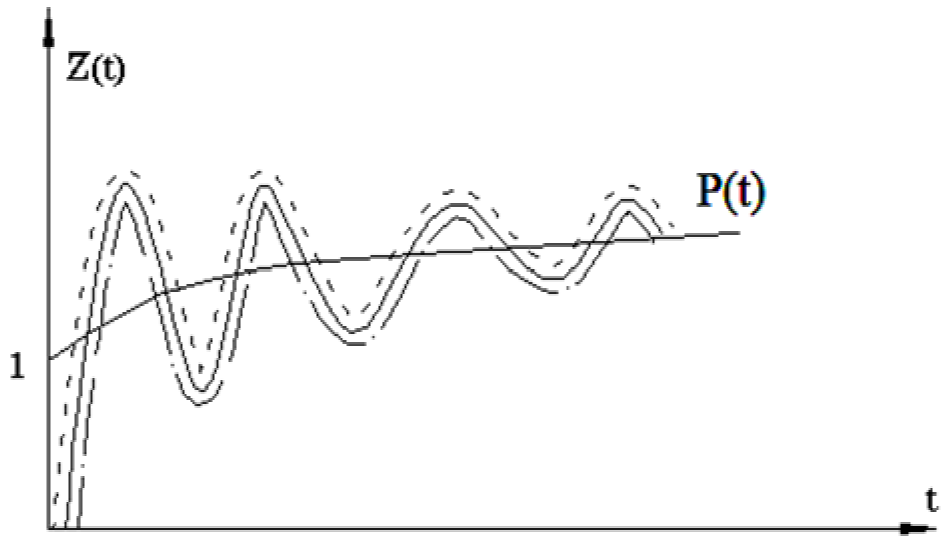

FIGURE 3. Vibrations of viscoelastic systems under constant external load

ANALYSIS OF THE RESULTS

To confirm the validity of qualitative solutions, we present a numerical solution of the IDE of the form

$$\ddot{z}(t) + \omega^2(1 - R*)z(t) = F(t), \quad (21)$$

under initial conditions

$$z(o) = 0, \dot{z}(o) = 0, \quad (22)$$

for a specific relaxation core

$$R(t-\tau) = \varepsilon e^{-\beta(t-\tau)}(t-\tau)^{\alpha-1}$$

and the source data

$$\varepsilon = 0.1, \quad \alpha = 0.125, \quad \beta = 0.05, \quad \omega = 2\pi, \quad q_0 = 2\omega^2.$$

The solution of IDE (21) under initial conditions (22) by a method based on the exclusion of weakly singular singularities of integral and IDE [1] has the form

$$z_n = \sum_{j=0}^{n-1} A_j(t_n - t_j)\left[F_j - \omega^2(z_j - \frac{\varepsilon}{\alpha}\sum_{k=0}^{j} B_k\, e^{-\beta t_k} z_{j-k})\right] n = 1,2,\ldots, \quad (23)$$

In particular, $F(t) = q_o$

$$z_n = \sum_{j=0}^{n-1} A_j(t_n - t_j)\left[q_0 - \omega^2(z_j - \frac{\varepsilon}{\alpha}\sum_{k=0}^{j} B_k e^{-\beta t_k} z_{j-k})\right]. \quad (24)$$

If the initial conditions for IDE (21) are set as

$$z(0) = 0, \dot{z}(0) = 1, \quad (25)$$

then

$$z_n = t_n + \sum_{j=0}^{n-1} A_j(t_n - t_j)\left[q_0 - \omega^2(z_j - \frac{\varepsilon}{\alpha}\sum_{k=0}^{j} B_k e^{-\beta t_k} z_{j-k})\right]. \quad (26)$$

Finally, if the initial conditions for IDE (21) are given as

$$z(o) = 1, \dot{z}(o) = 0, \tag{27}$$

then

$$z_n = 1 + \sum_{j=0}^{n-1} A_j(t_n - t_j)\left[q_0 - \omega^2 (z_j - \frac{\varepsilon}{\alpha}\sum_{k=0}^{j} B_k e^{-\beta t_k} z_{j-k}) \right], \tag{28}$$

where

$$A_0 = \frac{(\Delta t)}{2}; \qquad A_j = \Delta t; \quad j = \overline{1, n} - 1; \qquad t_n = n\Delta t; \quad z(t_n) = z_n;$$

$$B_0 = \frac{(\Delta t)^\alpha}{2}; \qquad B_k = \frac{(\Delta t)^\alpha}{2}\left[(k+1)^\alpha - (k-1)^\alpha\right]; \qquad k = \overline{1, j} - 1;$$

$$B_j = \frac{(\Delta t)^\alpha}{2}\left[j^\alpha - (j-1)^\alpha\right]; \qquad k = j \cdot$$

The numerical calculation results based on the addition theorem are shown in the figure (dashed lines), and according to algorithm (24) (dashed dotted lines). It can be seen that numerical calculations fully confirm the qualitative analysis of the solution (20).

Thus, the proposed approach allows not only to construct elementary calculation formulas for the desired solutions, but also to conduct a complete qualitative and quantitative analysis of natural and forced vibrations, which is very important for engineers when performing dynamic calculations of elements of thin-walled structures made of hereditarily deformable material.

Let us focus on solving equation (21) under initial conditions (25) and (27). In the ideally elastic case for F(t)=q-const, the corresponding solution has the form

$$z_{yn}(t) = \frac{1}{\omega}\sin\omega t + \frac{q}{\omega^2}(1 - \cos\omega t); \tag{29}$$

$$z_{yn}(t) = \cos\omega t + \frac{q}{\omega^2}(1 - \cos\omega t). \tag{30}$$

Let us graphically study the results of numerical calculation using formulas (26) and (28). A comparative analysis of the IDE solution (21), with exact solutions (29) and (30) corresponding to an ideally elastic problem, confirms that algorithms (26) and (28) give high accuracy in solving both an ideally elastic problem and a viscoelastic formulation of the problem.

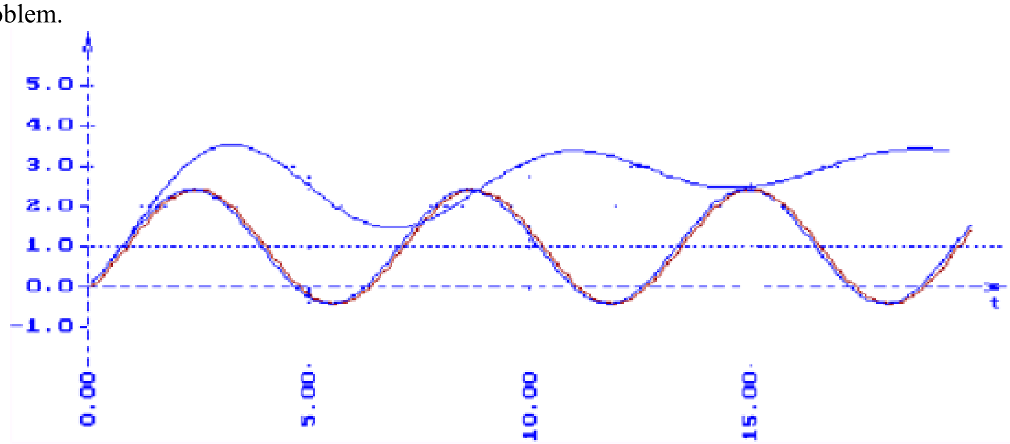

FIGURE 4. Qualitative analysis of elastic and viscoelastic vibrations

CONCLUSIONS

1. Taking into account the inherently deformable properties of the structural material lead to attenuation of free vibrations, while the attenuation rate significantly depends on the singularity parameter. The smaller the singularity parameter of the structural material, the higher the rate of attenuation of natural vibrations.

2. Vibrations of viscoelastic systems under the influence of constant external load pass near the curve of the creep function and fade over time along this curve.

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