Multi-parametric Mathematical Model of the Problem of Nonlinear Fluid Filtration in a Three-Layer Hydro-dynamically Connected Reservoir

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Abstract. The process of filtering highly nonlinear fluids through multilayer media has been investigated. A generalized mathematical model and a corresponding solution algorithm for a three-layer system have been developed. The model incorporates all previously proposed filtration laws commonly used in scientific research. Through an analysis of process similarity, these models and their computational algorithms were classified and grouped. They conform to the principles of modern computational technologies and have been tested using representative problem data.

INTRODUCTION

The physical and chemical phenomena occurring in porous media during the filtration of Newtonian and non-Newtonian fluids exhibiting nonlinear behavior have been extensively studied in works [1–6]. Various modeling approaches for describing such processes in specific porous media have been proposed.

Research in this field has contributed to the optimization of mathematical model construction through the development of multi-parameter mathematical models. An early attempt to introduce a multi-parameter law by assigning certain parameters to the filtration rate was made in [3]. However, these ideas were not further developed by other researchers, as the proposed law assumed uniqueness—each law corresponded to a single characteristic curve. The ambiguity in interpreting filtration laws was later resolved with the emergence of multi-parameter mathematical models [4–5].

In these studies, the development of multi-parameter functions enabled the construction of a comprehensive mathematical model that encompasses all known types of filtration laws, along with their respective initial, boundary, and interface conditions. The inclusion of parameters in boundary and interface conditions established a one-to-one correspondence between the governing equations and their associated conditions.

The resulting multi-parameter model facilitated the creation of parameterized multifunctional computational algorithms and corresponding mathematical software. Moreover, it allowed for the classification of models and numerical algorithms based on similarities in filtration laws and computational methods. This approach enables researchers to derive a boundary value problem relevant to a specific field directly from the general multi-parameter model—bypassing preliminary modeling stages. In practice, the user only needs to input field-specific data and perform the computational procedures.

All major stages of computational technology are aligned with the multi-parameter mathematical model as an integral part of the computational technology framework [7–8].

The aim of this work is to apply these developed methodologies to a multilayer system with hydrodynamic coupling, with a specific focus on a three-layer configuration.

METHOD

Let us assume that the filtration domain (Ω) consists of three subregions — (D_1, D_2 and D_3). The subregion is highly permeable, this indicates that the formation and fluid properties in the horizontal direction are roughly ten times more significant than those in the vertical direction.

Therefore, fluid flow within D_3 can be considered primarily horizontal. In contrast, the other two subregions, (D_1 and D_2), are poorly permeable; their vertical permeability is significantly greater than their horizontal permeability, which allows the assumption that the fluid motion within these layers occurs mainly in the vertical direction.

In addition, the bottom boundary of region D_2 is connected to the upper boundary of region D_3, forming a hydrodynamic coupling between them—that is, there is an exchange of fluid between these layers.

Let the entire domain be saturated with a non-Newtonian (anomalous or structured) fluid. At the onset of reservoir operation D_2, a disturbance arises within the filtration regions D_1 and D_3. As a result, fluid flows from region across the entire contact boundary into region D_2. The magnitude of this flow depends on the properties and velocity of the fluid, the intensity of extraction, as well as the filtration characteristics and heterogeneity of the formation boundaries.

The physical problem described above can be formulated mathematically as follows: it is required to determine the nontrivial unknown functions $U(x,\bar{z},t)$, $V(\bar{x},z,t)$, and the unfamiliar boundary functions $l_r(x,\bar{z},t), R_r(x,t)$, from the subsequent set of equations:

$$\left(hk(x)\chi\left(|\nabla u|/\beta\right)u_x/\mu\right)_x = M_2\left(x,t,u\right)u_t + a_1\left(V_1\right)_z\big|_{z=h_1} + a_2\left(V_3\right)_z\big|_{z=h_2} + f(x,t),\ x\in D_2, t>0 \qquad (1)$$

and

$$\left(Lk(z)\left(V_i\right)_z/\mu_i\right)_z = M_i\left(\bar{x},z,t\right)\left(V_i\right)_t,\ t>0, z\in D_i, \left(i=1,3\right). \qquad (2)$$

with initial conditions

$$u(x,\bar{z},0)=u_0(x,z),\ V_1(\bar{x},z,0)=V_2(\bar{x},z,0)=V_0(\bar{x},z) \qquad (3)$$

as well as conditions on natural borders

$$S_1\left(hk(x)\chi\left(|\nabla u|/\beta\right)/\left(\mu b\right)\right)u_x\big|_{x=x_0} + S_2 u\left(x,\bar{z},t\right)\Big|_{x=x_0} = \varphi_0(x_0,t), \qquad t>0, \qquad (4)$$

$$b_1\left(hk(x)\chi\left(|\nabla u|/\beta\right)/\left(\mu b\right)\right)u_x\big|_{x=x_L} + b_2\, u\left(x,\bar{z},t\right)\Big|_{x=x_L} = \varphi_2(x_L,t), \qquad t>0, \qquad (5)$$

conditions for unknown moving boundaries:

$$\begin{gathered} c_1\chi\left(|\nabla u|/\beta\right)\big|_{x=l_r^{\pm}} = 0,\ \ c_1\left(u(x,t)\big|_{x\in D} - u_0(x)\right)=0, \\ x=l_r^{\pm}\pm 0,\ r=\{0;\text{n}\};\ (r=0,\ x=l_0^{+}+0,\ r=n,\ \ x=l_n^{-}-0), \end{gathered} \qquad (6)$$

$$c_2\left(\|\nabla u\|_{x=l_r^{\mp}\mp 0} = \|\nabla u\|_{x=l_r^{\mp}\pm 0} = \beta\right),\ \ c_2\left(u(x,t)\big|_{x=l_r^{\mp}\mp 0} - u(x,t)\big|_{x=l_r^{\mp}\pm 0}\right)=0,\ \ r=\overline{0,n}. \qquad (7)$$

$$c_3\left(\|\nabla u\|_{x=l_r^{\pm}\mp 0} - \beta\right)=0,\quad c_3\left(u(x,t)_{x\in D} - u_0\right)=0, \qquad (8)$$

$$c_4\left(\|\nabla u\|_{x=R_{1\sigma}^{\pm}-0} = \beta_1; \|\nabla u\|_{x=R_{1\sigma}^{\pm}+0} = \beta_1\right),\ \ c_4\left(\|\nabla u\|_{x=R_{2\sigma}^{\pm}-0} = \beta_2; \|\nabla u\|_{x=R_{2\sigma}^{\pm}+0} = \beta_2\right), \qquad (9)$$

$$c_4\left(u(x,t)\big|_{x=R_{1\sigma}^{\pm}-0} = u(x,t)\big|_{x=R_{1\sigma}^{\pm}+0}\right),\ c_4\left(u(x,t)\big|_{x=R_{2\sigma}^{\pm}-0} = u(x,t)\big|_{x=R_{2\sigma}^{\pm}+0}\right), \qquad (10)$$

$$\begin{gathered} c_4\left(\|\nabla u\|_{x=R_{1r}^{\pm}-0} = \beta_1; \|\nabla u\|_{x=R_{1r}^{\pm}+0} = \beta_1\right),\ c_4\left(\|\nabla u\|_{x=R_{2r}^{\pm}-0} = \beta_2; \|\nabla u\|_{x=R_{2r}^{\pm}+0} = \beta_2\right), \\ r=\overline{1,n-1},\ \ \sigma=\{0,1\}. \end{gathered} \qquad (11)$$

$$c_4\left(u(x,t)\big|_{x=R_{1r}^{\pm}-0} = u(x,t)\big|_{x=R_{1r}^{\pm}+0}\right),\ \ c_4\left(u(x,t)\big|_{x=R_{2r}^{\pm}-0} = u(x,t)\big|_{x=R_{2r}^{\pm}+0}\right), \qquad (12)$$

including the upper and lower boundary constraints

$$\overline{\psi}_1 \cdot (V_1)_z\Big|_{z=h_0} = 0, \quad \overline{\psi}_2 \cdot (V_3)_z\Big|_{z=h_2} = 0, \quad t \geq 0. \tag{13}$$

In the equation (1)

$$\chi(\eta) = \frac{\left[\eta^{\lambda} - \gamma^{\lambda}(2-\lambda)\right]^{\theta} \cdot \eta^{-1} \cdot \beta^{\theta-1}}{\left\{1 + \left[(\lambda-\gamma)^{\lambda} + \eta^{\lambda}\right]^{\frac{1}{2}}\right\}^{\lambda-1}}. \tag{14}$$

is parametrized function. The parameter property (14) $\theta, \lambda, \gamma, \eta$ takes values as defined in Table 1, thereby determining the particular functional form corresponding to each fluid filtration law.

TABLE 1. The values of the parameters are determined by the filtration law adopted in the mathematical model.

laws	Ω	θ	λ	γ	η	c_1	c_2	c_3	c_4	Note
I	D_r	1	1	1	d_1	1	0	0	0	Law with an initial gradient [9].
	D	1	1	η	d_2	1	0	0	0	
II	D_r	1	1	μ_0	d_1	0	1	0	0	Polygon law [10].
	D	1	1	0	d_2	0	1	0	0	
III	D	1	2	1	η	0	0	0	0	Hyperbolic Law [10].
IV	D	1	2	2	η	0	0	0	0	Heeg's law [11].
V	D	1	1	0	η	0	0	0	0	Darcy's Law [10,12].
VI	D_r	1	1	ξ	d_1	0	1	0	0	Curvilinear law [4].
	D	1	2	1	d_2	0	1	0	0	
VII	D_r	θ	1	1	d_1	1	0	0	0	Law according to [12, 13].
	D	1	1	η	d_2	1	0	0	0	
VIII	D_r	1	1	0	d_1	0	0	1	0	Law according to [14].
	D	1	1	η	d_2	0	0	1	0	
IX	D_r	1	1	$\overline{\mu_0}$	d_1	0	1	0	0	A variant of the curvilinear law [4].
	D	1	1	2	d_2	0	1	0	0	
X	D	$\alpha+1$	1	0	η	0	0	0	0	Power law [12].
XI	D	1	2	2	η	0	0	0	0	Christanovich's law [15].
XII	D_1	1	2	2	g_1	0	0	0	1	Structured law [16,17].
	D_2	1	1	ξ_2	g_2	0	0	0	1	
	D_3	1	1	0	g_3	0	0	0	1	
XIII	D_1	1	2	1	η	0	0	0	1	Variant of the structured law [18].
	D_2	1	1	ξ_2	g_2	0	0	0	1	
	D_3	1	1	0	g_3	0	0	0	1	

The parameters e_1, e_2, e_3, e_4 are set by the condition at the mobile boundaries of outrages as well as at the boundaries of each zone, depending on the filtration laws applied, $\xi, \xi_1, \xi_2, \beta_1, \beta_2, \beta_3$ determines the coefficients at critical pressuregradients, in accordance with the filtration laws.
Here $g_1 = \eta\beta \;\; (\eta\beta > \beta_1)$, $\;d_1 = \eta \;(\eta < 1)$, $\;d_2 = \eta \;(\eta > 1)$, $\;g_2 = \eta\beta \;\; (\beta_1 < \eta\beta < \beta_2)$, $\;g_3 = \eta\beta \;\; (\eta\beta < \beta_1)$.

In problems (1) – (13), coefficients and parameters $\mu_1, \mu_2, \mu, k(x), k(z), \psi_1, \psi_2, \psi_3, \psi_4, M(x,t), M_i(z,t), f(x,t)$ are identified variables and functions that capture the characteristics of the medium and fluid [10-21], in addition

$$\Omega = D_1 \cup D_2 \cup D_3,\ D_2 = \overline{\Omega} - \sum_{r=0}^{n} \Omega_r,\ \Omega_r = \left\{x : l_r^- < x < l_r^+\right\},\ \overline{\Omega} = \left\{x : x_0 \le x \le x_L\right\}$$

$$D_1 = \left\{z : h_0 \le z \le h_1\right\},\ D_3 = \left\{z : h_2 \le z \le h_3\right\}.\ f(x,t) = a_0 \sum_{r=1}^{n-1} q_r \delta(x - x_r),\ S_1 + S_2 \ne 0, b_1 + b_2 \ne 0.$$

$$\Omega_1 = \left\{x : x_0 < x < R_{01}^+,\ \bigcup_r (R_{1r}^- < x < R_{1r}^+)\right\},\ \Omega_2 = \left\{x :\ \bigcup_r (R_{1r}^+ < x < R_{2r}^+;\ R_{2r}^- < x < R_{1r}^-)\right\},$$

$\Omega_3 = \left\{x :\ \bigcup_r (R_{2r}^+ < x < R_{2(r+1)}^-)\right\}$. $u = \left\{u_1; u_2; u_3\right\}$ if $x = \left\{\Omega_1; \Omega_2; \Omega_3\right\}$, respectively, for problems of structured fluids.

$\varphi_0(x_0, t)$ and $\varphi_2(x_2, t)$ it can take the values 0 of or q_0, depending on the reservoir physics being studied and the development conditions describing this boundary value problem [17,18,22].

Obviously, problem (1) – (14) is nonlinear; therefore, analytical solutions are not possible, and numerical iterative methods must be employed. This section describes, in a general and descriptive manner, the methodology for constructing computational algorithms via numerical methods; the specific algorithms are covered in a separate publication In this section, an iterative method is applied to the nonlinear terms of the equations, thereby linearizing them. Next, we introduce the flow function [20] and reformulate problem (1) – (14) in terms of this variable, converting it into a flow boundary value problem. This boundary value problem is then integrated with respect to a spatial variable over discrete segments $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, using the mean value properties of the integral. Analogously, the direct method is used for the time derivative. After integrating the flow expressions over the given segments $[x_{i-1}, x_i]$ and applying the necessary transformations and simplifications, we obtain a finite-difference boundary value problem for the flow [20–24].

In regions D_1 and D_2, integration is performed over their respective segments $[z_{j-\frac{1}{2}}, z_{j+\frac{1}{2}}]$, and the method of lines is also applied for domain by z for t.

Considering the relationship between the flow and the sought functions, formulas are derived for determining the run coefficients both along the OX, and OZ. -directions, as well as for the flow and the corresponding functions at the domain boundaries, based on the problem's boundary conditions. Expressions are also obtained for determining the positions of the unknown moving (perturbation) boundaries. The iterative method is further applied to the equations defined on these unknown boundaries, yielding the required relations.

A simplified algorithm for solving the finite-difference iterative system can be summarized as follows:

1. Initially, $\psi_1 = \psi_2 = 0$ the problem is solved in region D_2 without accounting for neighboring layers.
2. The column coefficients are computed in the forward direction for region $\overline{\Omega}$.
3. The positions of the unknown moving boundaries are determined using the boundary location method.
4. Using these newly found boundary positions as initial data, steps (1) – (3) are repeated iteratively.
5. Convergence conditions at the perturbation boundaries are assessed. Upon satisfaction

o The values of the sought functions (and, if necessary, the flow) are computed in the reverse direction, and the iteration criteria are verified for both the primary functions and the flow.

o If convergence is achieved, the computational procedure advances to the next time step; otherwise, the iteration cycle is repeated.

6. After each iteration step, the previously obtained values are used as the initial approximation for the next iteration.
7. The problem is then solved in domains D_1 and D_3 (i.e., by z -direction), the number of flow values is computed, and the procedure proceeds to the next spatial step, repeating the computational cycle.

The above computational sequence applies to problems with unknown moving boundaries. It should be noted that the original multi-parameter preliminary –boundary value problem (1) – (14) incorporates 13 various filtration legislations, also the construction of corresponding computational algorithms depends on the classification group to which a particular mathematical model belongs. The computational rules and specific algorithmic features vary for each group. Currently, no analogues of such a formulation for multilayer media exist in the scientific literature.

Analysis of these laws allows for a preliminary classification into three groups based on the similarity of their computational processes:

-Group I: Laws I, VII, and VIII — characterized by the presence of both disturbed and undisturbed zones.

-Group II: Laws II, VI, IX, XII, XIII — where the disturbance extends throughout the entire filtration region, with unknown moving boundaries separating different filtration zones.
-Group III: Laws III, IV, V, X, XI — which contain no unknown perturbation boundaries but exhibit individual nonlinear characteristics that must be accounted for.

The computational sequences for these three groups differ and possess their own distinct features. Accordingly, multi-parameter computational algorithms are constructed with consideration of these characteristics. For numerical implementation, a reference table (analogous to Table 1) has been developed, containing the coefficients and parameters selected according to their respective classification groups [25–32].

RESULTS AND DISCUSSIONS

To demonstrate the capabilities of the developed computational algorithms, representative calculation fragments for the following test data are presented;

$$q_0 = 0,0864,\ L = 4000м,\ k = 0,8,\ \Delta\tau = 0,00675;\ u_0 = 1, h = 25\cdot 10^{-3},\ \beta = 0,01,\ M = 1\ , V_0 = 1,$$

Assume that $x_0 = 0$ a well with a flow rate is located at the point $q_0 = 0$.

In the partial case for law VI , the calculation results are given in Table 2, which shows the function values for multiple t and x together with the corresponding positions of the perturbation boundary.

TABLE 2. Evolution of pressure and perturbation boundaries for the curved filtration law.

x \ t	10^{-2}	10^{-1}	$3\cdot 10^{-1}$	$7\cdot 10^{-1}$
0	0,97120	0,95317	0,93919	0,92504
0,1	0,98314	0,97212	0,95216	0,93112
0,5	0,99201	0,98315	0,97317	0,94624
0,9	0,99992	0,99612	0,98926	0,95566
1	0,99999	0,99817	0,98923	0,96224
$l(t)$	0,2344	0,5764	0,7312	0,8114

The table shows that due to the homogeneity of the well-permeable formation and the absence of cross-flows from neighboring formations, the pressure line is smooth. The disturbance boundary also slowly moves towards the boundary of the filtration region. To trace the rate of propagation of the boundaries of disturbances, between small and large areas of disturbance, the same problem was solved for $q_0 = 0$, and the well is located at the point $x_1 = 0,5$; with the corresponding flow $q_1 = 0,1718$ rate, the initial perturbation boundary $\overset{(0)}{l_1^-} = 0,4839$ (it is calculated using the formula $\overset{(0)}{l_1^-} = x_1 - \overset{(0)}{l_1^+} \cdot \frac{q_1}{2}$). The progression of the left and right boundaries is partially shown in Table 3, which makes it possible to visually trace the left and right borders.

TABLE 3. Numerical values of the movement of perturbation boundaries around the central well

t	l_1^-	l_1^+	t	l_1^-	l_1^+
$1\cdot 10^{-3}$	0,4214	0,5826	$1\cdot 10^{-2}$	0,2710	0,7300
$3\cdot 10^{-3}$	0,3721	0,6318	$3\cdot 10^{-2}$	0,1501	0,9502
$5\cdot 10^{-3}$	0,3325	0,6732	$5\cdot 10^{-2}$	0,0902	0,9101
$7\cdot 10^{-3}$	0,3014	0,7001	$7\cdot 10^{-2}$	0,0301	0,9711
$9\cdot 10^{-3}$	0,2815	0,7211	$9\cdot 10^{-2}$	0,0000	1,0000

TABLE 4. Dynamics of changes in perturbation boundaries around three wells

t	l_1^-	l_1^+	l_2^-	l_2^+	l_3^-	l_3^+
$6,7\cdot10^{-4}$	0,224	0,276	0,474	0,526	0,724	0,776
$1,3\cdot10^{-3}$	0,214	0,286	0,464	0,536	0,714	0,786
$3,3\cdot10^{-3}$	0,194	0,306	0,443	0,556	0,699	0,806
$6,1\cdot10^{-3}$	0,174	0,325	0,424	0,575	0,674	0,825

Table 4 shows an overview of the dynamics of moving the boundaries of disturbances around three wells. Table 5 provides the function's average values $u(x,t)$ for separately selected filtering laws in the integral sense.

TABLE 5. Averaging the pressure value in the vicinity of the source of disturbance for various laws.

x / **laws**	$675\cdot10^{-5}$	$135\cdot10^{-4}$	$331\cdot10^{-4}$	$675\cdot10^{-4}$
I	0.98132	0.96289	0.86187	0.81830
II	0.98616	0.96721	0.87530	0.83975
III	0.98919	0.97066	0.87850	0.83280
V	0.99016	0.97185	0.87996	0.83415
VII	0.98502	0.96634	0.87160	0.82670

Table 6. shows the results of calculating the function values in the left part of the reservoir, when the source is $x=0,5$ situated at the point e, and $\tau=10^{-1}$. The final results of calculations allow us to conclude that the curve of change of the function for laws IX, IV, III lies between V and I, and at the same time the minimum pressure will be at the bottom of the source.

TABLE 6. Values of pressure in the left part of the filtration zone, where the source occupies the central region of the reservoir

x	**0**	**0,1**	**0,2**	**0,3**	**0,4**
I	1,0000	0,9901	0,9012	0,7870	0,6846
III	0,9143	0,8884	0,8486	0,7946	0,7388
IV	0,8923	0,8737	0,8314	0,7916	0,7493
V	0,8095	0,8012	0,7911	0,7887	0,7614
IX	0,9731	0,9325	0,8825	0,8183	0,7821

Based on the calculation results, the function's change for laws III, IV, and IX is intermediate between laws V and I, with the lowest pressure at the base of the source.

Table 7 $\beta=0,5\cdot10^{-4};10^{-3}$, $m=0,2$, $\mu=0,3$, $x=0,005$, $m=0,18$, $\alpha=1$, $b=5a$, $h_1=80$, $u_0=1$, $q=50$ shows the pressure change in the region D_1 and D_2 at the point $x=0$ and the value of the flow from D_2 to D_1.

TABLE 7. Temporal and spatial dynamics of the integral pressure function in the region D_2 and also for the flow of fluids from D_1 at different β temperatures

t	$\beta = 0$	$\beta = 10^{-5}$	$\beta = 10^{-3}$
0,01	0,99250	0,98602	0,93251
0,05	0,96361	0,94212	0,90212
1	0,93210	0,91322	0,88214
0,15	0,90316	0,88220	0,85675

TABLE 8. Value of the flow rate between layers

t	$\beta = 0$	$\beta = 10^{-5}$	$\beta = 10^{-3}$
0,01	0,13254	0,11223	0,09621
0,05	0,19425	0,15231	0,12321
0,1	0,22321	0,18013	0,15431
0,15	0,26255	0,21860	0,18222

CONCLUSION

Results from computational experiments indicate that the proposed multi-parameter model for a three-layer reservoir, which accounts for all established fluid-filtration laws, provides a concise and practical representation of the associated mathematical models. Its algorithms can be applied to evaluate technical and economic performance in multilayer porous media.

The presented three-parameter model is novel in the study of fluid-filtration processes in multilayer systems—both in terms of the problem formulation and in the development of computational algorithms for its solution. Since the model encompasses a wide range of filtration problems for three layer reservoirs, its potential and accuracy should be further assessed using three-polar computational techniques.

It is worth noting that, during the iterative determination of the left and right boundaries of the disturbance zone caused by a point source (for instance, following Law I with an initial gradient), a two-sided ('shuttle') iteration way [4, 22, 30] can be used. This approach enables rapid determination of boundary positions and allows for a more precise estimation of the fluid flow magnitude from regions D_3 and D_1 to region D_2.

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