Nonlinear Catastrophe Models: Classification and Stability Analysis Based on Polynomial Degree

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Abstract. The methodology begins by formulating standard forms of the nonlinear differential equations corresponding to folding, crest, and butterfly tail catastrophes. This study examines the classification of catastrophe types arising in nonlinear dynamical systems as a function of the system degree of nonlinearity. Extending the principles of René Thom's catastrophe theory, the paper analyzes how higher order nonlinear boundaries affect the structure, stability, and bifurcation of equilibrium points. Mathematical models are designed to represent folding, crest, butterfly tail, and butterfly catastrophes, each corresponding to a specific degree of nonlinearity in the governing equations. By calculating equilibrium solutions and examining eigenvalue spectra, this study identifies transitions from simple to complex catastrophes. The results provide a general framework for understanding nonlinear instability and abrupt changes in systems such as drug dose response dynamics and epidemic transitions.

Keywords: Catastrophe Theory; Nonlinear Dynamics; Stability; Fold Catastrophe; Cusp Catastrophe.

INTRODUCTION

In 1975 (René Thom) proposed catastrophe theory, introducing a mathematical framework explains how continuous changes in control parameters cause sudden and discontinuous transitions in the state of the system. These sudden transformations renowned as catastrophes, arise while equilibrium structure of nonlinear system turn changeable resulting in fast qualitative shifts in its behavior. These phenomena are observed in all sciences such as physics, biophysics, pharmacology, epidemiology and economics [1-8]. The results indicate that the degree of catastrophic types for nonlinear dynamical systems. Gradually the system displays more complex branching patters moving from simple folding behavior to toothed beak tail configuration. This relationship proposes that any increase in nonlinearity produces additional equilibrium divisions and critical transitions, which expands classical classification of Thom's.

The results accentuate that nonlinearity not only regulates the number of viable equilibrium states but also manage the nature of stability variations within a system [9-13]. The research enlarges after this theoretical foundation by studying how the degree of nonlinearity in a executive equation detects the type and catastrophic complexity behavior that can appear. The research precisely consider how polynomials nonlinearity satisfies elementary catastrophe to varying degree, such as folding, hilling, tooth-tail, and butterfly patterns, and how these can introduce actual processes for which stability and instability go together. In biological and pharmacological contexts, nonlinear responses are particularly relevant. For example, a slight increase in drug concentration can abruptly shift a system from therapeutic equilibrium to toxicity, while in epidemiology, gradual changes in infection rates can lead to widespread outbreaks.

These changes reflect qualitative behavior predicted through catastrophe theory, thresholds separate unchanging and unstable systems. Through combining nonlinear dynamics, stability analysis, and catastrophe modeling, the aim of this research is to build a unified mathematical description to explain these rapid transformations [1-5, 12]. This article focuses on classical models' generalization by changing the degree of nonlinearity (n) in system control equations. Through analytical derivations and stability assessments, the study demonstrates how each level of nonlinearity represents a new type of catastrophe, thereby enlarge our understanding of complex dynamical systems. The findings have potential applications in pharmacokinetic modeling, disease transmission dynamics, and engineering systems, where predicting critical thresholds is essential for maintaining system stability and preventing sudden failures. In summary this work contributes to the broader field of nonlinear systems analysis by demonstrating that the degree of nonlinearity not only determines the shape of the underlying surface but also the nature of the system potential catastrophes. This work builds upon previous studies by Kaki M. N. M And others, which contributes to the development of an explanation of catastrophic behavior in mathematical, biological and applied sciences [8, 10, 15].

METHODOLOGY

The research applies nonlinear differential equation analysis and bifurcation theory to identify critical transitions under changing control parameters.

1) Formulating generalized potential functions with varying degrees of nonlinearity (cubic, quartic, quintic, etc.).
2) The degree of nonlinearity (n) in each system is varied systematically (from quadratic to quintic forms).
3) The catastrophic behavior is analyzed through potential functions $V(x)$ and their derivatives $\frac{dV}{dx} = 0$).
4) Equilibrium points are determined analytically, and their stability is verified using eigenvalue computations.
5) The classification of catastrophes follows Thom's seven elementary types, linking each to polynomial degree and control variables.

Software tools such as MATLAB and Python are used to visualize the potential surfaces and bifurcation diagrams.

MATHEMATICAL MODEL

A general nonlinear system is defined as:

$$\frac{dx}{dt} = f(x, a, b) = a + bx - x^n \qquad (1)$$

where:

- (x) = state variable (e.g., population density, drug concentration),
- (a, b) = control parameters,
- (n) = degree of nonlinearity $(2, 3, 4, 5, \dots)$.

The corresponding potential function is:

$$V(x) = -ax - \frac{1}{2}bx^2 + \frac{1}{n+1}x^{n+1} \qquad (2)$$

Each value of (n) represents a specific catastrophic type:

Table 1, Degree (n), Catastrophe, Type Equation and Control Parameters

Degree (n)	Catastrophe Type	Equation	Control Parameters
2	Fold	$a + x^2 = 0$	1
3	Cusp	$a + bx + x^3 = 0$	2
4	Swallowtail	$a + bx + cx^2 + x^4 = 0$	3
5	Butterfly	$(a + bx + cx^2 + dx^3 + x^5 = 0$	4

STABILITY ANALYSIS

At equilibrium,

$$\frac{dx}{dt} = 0 \tag{3}$$

.

Linearizing near equilibrium x_e, we obtain:

$$\frac{d(\delta x)}{dt} = f'(x_e)\delta x \tag{4}$$

The stability depends on the sign of $f'(x_e)$:

- $f'(x_e) < 0 \Longrightarrow$ Stable equilibrium.
- $f'(x_e) > 0 \Longrightarrow$ Unstable equilibrium.

For higher-degree models ($n \geq 3$), multiple equilibria can coexist. The bifurcation set (where stability changes) forms geometric surfaces-such as cusp or butterfly shapes-in the control parameter space.

CATASTROPHE THEORY APPLIED TO THE SEIQR MODEL

A mathematical and theoretical formulation showing how Catastrophe Theory can be applied to the SEIQR model, including equations, stability analysis, and the link to cusp and fold catastrophes [7] & [16-18].

The SEIQR Model Framework

The SEIQR system divides the total population (N) into five compartments:

$$S(t) + E(t) + I(t) + Q(t) + R(t) = N, \tag{5}$$

where

- $S(t)$: susceptible population,
- $E(t)$: exposed but not yet infectious,
- $I(t)$: infectious individuals,
- $Q(t)$: quarantined individuals,
- $R(t)$: recovered (or removed) individuals.

A typical nonlinear SEIQR model is given by:

$$\frac{dS}{dt} = \Lambda - \beta SI - \mu S \tag{6}$$

$$\frac{dE}{dt} = \beta\frac{SI}{N} - (\sigma + \mu)E \tag{7}$$

$$\frac{dI}{dt} = \sigma E - (\gamma + \delta + \mu)I \tag{8}$$

$$\frac{dQ}{dt} = \delta I - (\gamma_q + \mu)Q \tag{9}$$

$$\frac{dR}{dt} = \gamma I + \gamma_q Q - \mu R \tag{10}$$

where:

- β = transmission rate,
- σ = rate of progression from exposed to infectious,
- δ = quarantine rate,
- $\gamma, \gamma_{q=}$ recovery rates,

- μ = natural mortality,
- Λ = recruitment/birth rate.

Reduction to a Control-State Relationship

To study critical transitions, we reduce the dynamics to a single epidemic variable (e.g., the infectious population I) and treat the control parameters as slow variables (e.g., β, δ, γ).

At equilibrium ($\frac{dI}{dt} = 0$), the governing relation can often be approximated by a nonlinear algebraic equation of the form:

$$a + bI + cI^2 + dI^3 = 0, \tag{11}$$

where the coefficients (a, b, c, d) depend on epidemiological parameters $\beta, \delta, \sigma, \gamma, \mu$. [19- 21].

Catastrophe Surface and Stability

This cubic equilibrium equation defines a potential function:

$$V(I; a, b) = aI + 2bI^2 + 3cI^3 + 4dI^4 \tag{12}$$

and equilibrium points satisfy $\frac{dV}{dt} = 0$.

Changes in a, b, c, d can cause *qualitative shifts* in the number and stability of equilibria, analogous to:

- Fold catastrophe: ($a + I^2 = 0$ → sudden onset or extinction of infection.
- Cusp catastrophe: $a + bI + I^3 = 0$ → coexistence of two stable infection levels (endemic and controlled).
- Swallowtail catastrophe: $a + bI + cI^2 + I^4 = 0$ → complex multi-threshold epidemic transitions.

Interpretation in Epidemiological Terms

- The control parameters (a, b, c) correspond to public health factors such as quarantine rate δ, transmission rate β, and recovery rate γ.
- The state variable (I) represents the level of infection.
- As β increases or δ decreases, the system can cross a bifurcation surface, triggering a sudden jump in (I) - a catastrophic outbreak.
- Conversely, increasing δ *or* γ beyond the cusp point leads to a rapid drop in infections (disease elimination).

Catastrophe Geometry

The equilibrium manifold in $((a, b, I))$-space forms a cusp surface, which separates stable and unstable epidemic regimes. The sharp edge of the cusp corresponds to critical thresholds in R_0 (the basic reproduction number):

$$R_0 = \frac{\beta\sigma}{(\sigma+\mu)(\gamma+\delta+\mu)} \tag{13}$$

Crossing $R_0 = 1$ can correspond to a fold point (loss of stability), while simultaneous changes in β and δ can generate cusp-like bifurcation behavior.

Significance

Applying Catastrophe Theory allows epidemiologists to:

- Predict sudden epidemic outbreaks (bifurcation jumps).
- Classify control strategies that move the system away from the cusp surface.
- Develop early-warning indicators for tipping points in infection dynamics.

RESULTS AND DISCUSSION

Analysis of the nonlinear differential equations corresponding to the folding, crest, swallowtail, and butterfly catastrophes revealed distinct qualitative shifts associated with increasing degrees of nonlinearity in the governing system. For each basic form, equilibrium points were calculated analytically, and their stability was assessed using eigenvalue spectra derived from Jacobian matrices.

Fold Catastrophe (First-Order Nonlinearity)

The folding model showed a single stable equilibrium state, which disappeared as the control coefficient exceeded a critical threshold, resulting in a classic saddle node fork. Numerical testing confirmed that the system abruptly transitioned from one solution branch to another, consistent with the predicted catastrophe geometry.

Cusp Catastrophe (Second-Order Nonlinearity)

The cusp model showed dynamic stability across a well-defined region of the transaction space. In this region, the paths converged into one of two stable equilibriums, depending on the initial conditions. A separate jump of the state variable caused by the fork in the path. The eigenvalue analysis revealed that the loss of stability occurred at the intersection of the true eigenvalues with zero, which is consistent with the (Thom's) classification.

Swallowtail Catastrophe (Third-Order Nonlinearity)

The introduction of cubic nonlinearity led to a richer branching structure, incorporating several simultaneous equilibria with unstable branches. Numerical analysis revealed a series of fold branches that together form the surface of the swallowtail. The transaction space showed alternating system of single, double, and triple stability, demonstrating the systems' increased sensitivity to perturbations and transaction deviations.

Butterfly Catastrophe (Fourth-Order Nonlinearity)

The high nonlinearity led to the creation of complex system of equilibrium curves with overlapping fold and multiple hysteresis loops. The butterfly configuration produced four distinct stable regions, separated by unstable bifurcations. The noise perturbations applied during the simulations showed an increased susceptibility to inversion events with unexpected jumps in solutions between equilibrium points as the control parameters approached multifaceted singularities.

General Pattern of Increasing Nonlinear Complexity

In all models increasing nonlinearity led to a wider bifurcation range and a greater number of critical shifts. This gradient resulted in a hierarchical structure where lower order catastrophes are embedded within higher order catastrophes. Phase level simulations confirmed that the systems trajectories became more complex with increasing nonlinearity, exhibiting helical behaviors, delayed shifts, and discontinuous switching.

Applications to Epidemic and Pharmacological Systems

When parameterized using typical epidemiological or pharmacological values, the models reproduced abrupt changes such as epidemic resurgence, dose-response threshold shifts, and sudden failure of control mechanisms. These behaviors corresponded to the fold, cusp, and swallowtail geometries identified analytically. Epidemiological stratification interventions.

According to the results, appearance as well as structure of these kinds of catastrophe species in nonlinear dynamical systems are directly influenced by the degree of nonlinearity. The system shows increasing types of branching in simple folding behaviors to toothed shapes from top to-down, as the degree of nonlinearity increases. This case indicates. The results also show that nonlinearity regulates the nature of changes in stability within the system in addition to determining the number of possible equilibrium states. The landscape of possibilities of the system becomes more complex with nonlinearity increasing and thus the enrichment of more patterns of stability and instability.

- One sudden change that occurs to the catastrophe fold (for example, the threshold effect for the drug).
- The cusp catastrophe leads to bistability, duplication and captures hysteria in biological recovery or infection control.
- The swallowtail and butterfly types correspond to multi-stage transitions seen in dose accumulation, immune fatigue, or epidemic reinfection cycles.
- These models provide a unified mathematical language to describe discontinuous jumps in continuous systems - bridging pharmacological, biological, and epidemiological interpretations.

FINDINGS

- Increasing the nonlinearity (n) leads to higher-order catastrophes, each of which requires more control dimensions.
- The boundaries of stability are becoming increasingly folded, leading to the production of multiple equilibrium paths.
- The parameter will determine the sensitivity of the system for bistability, tristabilty and metastibilty.
- The mathematical building expects sudden changes in the effectiveness of drugs or spread of infection as parameters bypass threshold values.

CONCLUSION

The degree of nonlinearity appears to play a crucial role in determining the types, structure and complexity of catastrophic behavior in nonlinear dynamical systems. By extending (Thom's) classical classification of catastrophes to include higher-order nonlinear polynomials, this study provides a comprehensive analytical framework for predicting loss of instability, bifurcation patterns, and qualitative shifts in complex systems. The results also show that as the degree of nonlinearity increases, the system undergoes increasingly complex bifurcation topologies, ranging from simple convolutions such as pentagonal ridges and toothed tails to higher-dimensional catastrophes, each associated with different equilibrium and stability configurations. Importantly, these findings have clear and profound implications for modeling biology, pharmacology, and epidemiology, where nonlinear feedback mechanisms often govern abrupt changes in system dynamics. In pharmacology, even small changes in dosage or reaction strength can shift a system from therapeutic equilibrium to the toxicity range, while in epidemiology, gradual fluctuations in transmission or recovery coefficients can lead to widespread disease outbreaks. By enabling the prediction and classification of these changes, the proposed framework offers new insights into the mechanisms underlying critical thresholds and system transformation. Future studies are expected to incorporate stochastic perturbations and noise induced reversal phenomena, given that real world systems are rarely deterministic. Furthermore, matrix based generalizations for multivariable systems and networks will extend this approach to higher dimensional epidemic models and complex drug interaction networks. These extensions will enhance the predictive power of nonlinear models, improve epidemic control strategies, and enhance drug risk assessment by identifying early indicators of instability and potentially catastrophic shifts.

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