**Difference in topological characteristics between Cantor middle-third set and Sierpin´ski carpet**

Akihiko Kitada,1 Shousuke Ohmori,2, 3 Yoshihiro Yamazaki,1, 4 and Tomoyuki Yamamoto1, 4, 5, a)

1. *Insitute of Condensed-Matter Science, Comprehensive Research Organization, Waseda University,*

*3-4-1 Okubo, Shinjuku, Tokyo 169-8555, Japan*

1. *National Institute of Technology, Gunma College, 580 Toba-cho, Maebashi, Gunma 371-8530, Japan*
2. *Waseda Research Institute for Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku, Tokyo 169-8555, Japan*
3. *Faculty of Science and Engineering, Waseda University,*

*3-4-1 Okubo, Shinjuku, Tokyo 169-8555, Japan*

1. *Kagami Memorial Research Institute for Materials Science and Technology, Waseda University, 2-8-26 Okubo, Shinjuku, Tokyo 169-0051, Japan*

**Abstract.** A difference in topological nature between Cantor middle-third set and Sierpin´ski carpet, both of which are perfect, compact and self-similar spaces has been made clear through the investigations of the procedure of construction of an interesting quotient space. In spite of the similarity of the mathematical procedure of construction, there exists a fundamental difference in topological nature between them, and this difference affects the methods of construction of an interesting non-trivial quotient space of them. The totally disconnected nature (or, more generally, zero-dimensional) enables Cantor middle-third set to have a non-trivial quotient space which is self-similar. On the other hand, concerning Sierpin´ski carpet, because of the connectedness of its structure, no non-trivial quotient space which is self-similar can be constructed by such an elegant procedure as that for Cantor middle-third set. Various topologically signiﬁcant nature speciﬁc to Cantor middle-third set owe mainly to the totally disconnected nature of the set.

# INTRODUCTION

Cantor middle-third set (CMTS) and Sierpin´ski carpet (SC) are well known as typical fractal structures, both of which are self-similar, perfect, compact metric spaces. Since the procedure of the block construction or the coarse graining of a space corresponds mathematically to that of the construction of a quotient space of a space, nonlinear science researchers have been interested in a concept of quotient spaces[1,2].

In the present paper, a difference in topological nature between CMTS and SC, both of which are perfect, compact and self-similar, is discussed from a viewpoint of a general topology. In fact, owing to the zero-dimensional (0- dim) [3] or totally disconnected nature of CMTS, some interesting non-trivial quotient space of CMTS such as the space with a self-similarity can be constructed in an elegant way. Contrary to this, such an interesting non-trivial quotient space is hardly constructed for SC at least in the same elegant procedure with that for CMTS because of the connectedness of SC. In the following sections, this fact will be conﬁrmed.

# DEFINITIONS AND STATEMENTS

Some deﬁnitions and statements which will be used in this paper are summarized in this section.

# Zero-dimension and perfectness

A topological space (*Z,* τ) is said to be zero-dimensional (0-dim) provided that at every point *p* ∈ *Z* and for every

*U* ∈ τ containing *p*, there exists *u* ∈ τ ∩ ℑ (ℑ denotes the set of all closed sets of *Z*, that is, *u* is an open and closed

a)Electronic mail: Corresponding author: [tymmt@waseda.jp](mailto:tymmt@waseda.jp)

set) such that *p u U* . In any compact T2-space, the notion of 0-dim coincides with that of the totally disconnected nature [4] and it is easy to check that any 0-dim T0-space is totally disconnected.

∈ ⊂

A topological space (*Z,* τ) is said to be perfect provided that a singleton *p* , *p Z* is not τ-open, that is, any non- empty open set of (*Z,* τ) has at least two points. Any non-empty open set of a perfect space is perfect as a subspace. Both 0-dim and perfect are topological properties, that is, 0-dim and perfect are invariant under the homeomor-

{ } ∈

phism[5].

# Self-similarity

A topological space (*A,* τ) is self-similar provided that i) it is metrizable, that is, τ is identical with a metric topology

τ*d* deﬁned by a metric *d* on *A*. ii) there exists a set of contractions { *f j* : (*A,* τ*d* ) → (*A,* τ*d* ); *j* = 1*,..., m* (2 ≤ *m <* ∞)}

*m*

∪

such that *f j*(*A*) = *A*.

*j*=1

Especially, we call a metric space (*A,* τ*d* ) self-similar provided that the condition ii) is satisﬁed with respect to this metric *d*.

A sufﬁcient condition for a topological space to be self-similar is given in the following statement.

**Statement.** *The existence of a self-similar space which is homeomorphic to* (*Y,* τ) *is sufﬁcient for a topological space*

(*Y,* τ) *to be self-similar.*

(Concerning the proof, see Appendix 1).)

# Statement for contractions

*Let* (*Z,* τ*d* ) *be a compact metric space and the sytem* { *f j* : (*Z,* τ*d* ) → (*Z,* τ*d* )*, j* = 1*,..., m*}

*be a set of contractions d*( *f j*(*z*)*, f j*(*z* )) ≤ α *j*(η)*d*(*z, z* ) for *d*(*z, z* ) *<* η*,* 0 ≤ α *j*(η) *<* 1*,* infη*>*0 α *j*(η) *>* 0*,*

*j* = 1*,* · · · *, m* (2 ≤ *m <* ∞)*. m*

∪⊂

*Then, there exists a non-empty compact set X Z such that f j*(*X* ) = *X. This self-similar set X is unique in the set*

*j*=1

*of all non-empty compact sets of Z.*

*If the system of contractions* { *f j*} *satisﬁes following two conditions*

1. *Each f j is one to one,*

*m*

∪ { ∈ }

1. *The set z Z*; *f j*(*z*) = *z is not a singleton,*

*j*=1

*then the self-similar compact set X is perfect.*

*Furthermore, if the system of contractions* { *f j*} *satisﬁes the additional third condition*

*m*

∑ *in f*η*>*0α *j*(η) *<* 1*,*

*j*=1

*the self-similar compact set X is not only perfect but 0-dim.*

(Concerning the proof, see Appendix 2).)

# Decomposition space

A quotient space of a space *X* is deﬁned by a classiﬁcation of all points in *X* through the identiﬁcatioin of the different points based on an equivalence relation.

If *f* : (*X,* τ) → (*Y,* τ ) is a continuous, onto, closed map (A map *f* is said to be a closed map provided that for any

closed set *K* in *X* , *f* (*K*) is a closed set in *Y* .), then *h* : (*Y,* τ ) → (*Df ,* τ(*Df* ))*, y* → *f* −1(*y*) is a homeomorphism. Here

the set *Df* is { *f* −1(*y*) ⊂ *X* ; *y* ∈ *Y* } and the topology τ(*Df* ) is {*U* ⊂ *Df* ; ∪ *U* = ∪ *D* ∈ τ*,* each *D* ∈ *Df* is *f* −1(*y*)

for some *y*

*Y* . Topological space τ

is a quotient space of *X* τ

*D*∈*U*

∈ } (*Df ,*

(*Df* )) (

*x* ∼ *x* def *f* (*x*) = *f* (*x* )

=

*,* ) based on the equivalence relation ∼

Sometimes, the quotient space (*Df ,* τ(*Df* )) is called a decomposition space of (*X,* τ) due to *f* .

In general, a quotient space (*D,* τ(*D* )) of (*X,* τ) where *D* = {{*x*}; *x* ∈ *X* } ({*x*} denotes a singleton) and τ(*D* ) = {*U* ⊂

∪ ∈ τ}

*D* ; *U* is called the trivial quotient space. The trivial quotient space *D* of *X* is homeomorphic to *X* . Therefore, if *X* is self-similar, from the statement in B., *D* must be self-similar.

# Statement for partition

*Let* (*Z,* τ) *be a 0-dim, perfect T*0*(necessarily T*2*)-space. Then, there exists a partition Y, Z Y of Z where Y*

{ − } ∈

(τ ℑ) ϕ *and Y* ⫋ *Z. Here the subspace* (*Y,* τ*Y* ) *is 0-dim and perfect.*

∩ −{ }

(Concerning the proof, see Appendix 3).)

# Statement for connectedness

*Let* (*X,* τ) *be a compact normal space and* (*Xi,* τ*Xi* )*, Xi* ∈ ℑ−{ϕ}*, i* ∈ **N** *be connected subspaces of* (*X,* τ) *such that Xi*+1 ⊂ *Xi, i* ∈ **N***. Then, E* = *Xi is not empty and the subspace* (*E,* τ*E* ) *of* (*X,* τ) *is compact and connected* [5].

∩

*i*∈**N**

# CMTS AND SC

2

∪

CMTS is a subspace of closed interval [0*,* 1] such that CMTS= *f j*(CMTS) where the contractions *f*1(*x*) and *f*2(*x*)

*j*=1

are *x/*3 and *x/*3 + 2*/*3*, x* [0*,* 1], respectively. The sum of two contraction coefﬁcients is 2*/*3 *<* 1. Any metric space, which is 0-dim, perfect and compact, is homeomorphic to CMTS [6].

∈

8

× ∪

SC is a subspace of square [0*,* 1] [0*,* 1] such that SC= *f j*(SC) where

*j*=1

*f*1(*x*1*, x*2) = (*x*1*/*3*, x*2*/*3)*, f*2(*x*1*, x*2) = (*x*1*/*3 + 1*/*3*, x*2*/*3)*,*

*f*3(*x*1*, x*2) = (*x*1*/*3 + 2*/*3*, x*2*/*3)*, f*4(*x*1*, x*2) = (*x*1*/*3*, x*2*/*3 + 1*/*3)*, f*5(*x*1*, x*2) = (*x*1*/*3 + 2*/*3*, x*2*/*3 + 1*/*3)*, f*6(*x*1*, x*2) = (*x*1*/*3*, x*2*/*3 + 2*/*3)*, f*7(*x*1*, x*2) = (*x*1*/*3 + 1*/*3*, x*2*/*3 + 2*/*3)*, f*8(*x*1*, x*2) = (*x*1*/*3 + 2*/*3*, x*2*/*3 + 2*/*3)*,*

(*x*1*, x*2) [0*,* 1] [0*,* 1]. The sum of eight contraction coefﬁcients is 8*/*3 1.

∈ × ≥

In general, the statement C. holds, thus, SC is perfect and compact, whereas CMTS is perfect, compact and 0-dim.

# DISCUSSIONS

**Quotient space for CMTS**

Since CMTS is a 0-dim, perfect, compact metric space as in the statement C., from the statement E., there exists a 0-dim, perfect, compact subspace (*Y,* τ*dY* ) of CMTS such that *Y* ⫋ CMTS. For example, *Y* =CMTS [0*,* 1*/*3]. Here the notation *dY* denotes the restriction of the metric *d* (*d*(*x, x* ) = *x x* ) on *Y* . Now, let *f* : (CMTS,τ*d* ) (*Y,* τ*dY* ) be a not one to one continuous, onto map. The simple one

| − | →

∩

{*f* (*x*) =

*x , x* ∈ *Y*

*q , x* ∈ CMTS −*Y*

is an example. Here *q* is an arbitrarily chosen point of *Y* . Since the singleton *q* is a closed set in a T1-space, it is easy to see that the map *f* is a continuous, closed map from CMTS onto *Y* . From D., *h* : (*Y,* τ*dY* ) (*Df ,* τ(*Df* ))*, y f* −1(*y*) is a homeomorphism. Here, (*Df ,* τ(*Df* )) is a quotient space of CMTS due to the above map *f* . Since (*Y,* τ*dY* ) is a 0-dim, perfect, compact metric space, it is homeomorphic to CMTS. Then, the quotient space *Df* is also homeo- morphic to CMTS. According to the statement in B., *Df* is self-similar. It must be noted that this *Df* is not the trivial quotient set *x* ; *x* CMTS because *f* is not one to one. In fact, for example, if *Y* =CMTS [0*,* 1*/*3], the set *q* (CMTS [2*/*3*,* 1]) is a point of *Df* . Namely, there exists a self-similar non-trivial quotient space of CMTS. Fur- thermore, since *Df* is homeomorphic to CMTS, it is a self-similar, 0-dim, perfect, compact metric space (Concerning the metrizability of *Df* , see Appendix 1).). Then, we can obtain a self-similar quotient space of *Df* which is again a 0-dim, perfect, compact metric space. Continuing this process, we attain an inﬁnite sequence {*Di*; *i* = 0*,* 1*,* 2*,* · · ·} in which *D*0 =CMTS and each *Di* is a quotient space of *Di*−1

→ →

{ }

{ } ∪ ∩

{{ } ∈ } ∩

# Quotient space for SC

According to the statement C., SC is perfect and compact. CMTS is also perfect and compact, and furthermore, 0-dim.

∩

In this paragraph, using F., we will demonstrate that SC is a connected space. SC is given by the intersection *Xi*

*i*∈**N**

of closed sets *Xi* = ∪ *f j*1 ◦ · · · ◦ *f ji* ([0*,* 1] × [0*,* 1])*, j*1 ∈ {1*,...,* 8}*,..., ji* ∈ {1*,...,* 8}*, i* ∈ **N** in [0*,* 1] × [0*,* 1]. Here

each *X*

*j*1··· *ji*

*X X* . Fig.1 shows, for example, *X*

with 9 windows.

*i* has ﬁnite number of windows and obeys the relation

*i*+1 ⊂ *i* 2

As shown in Fig.1, any two points *p* and *q* are joined by an arc in *X*2. Therefore, *X*2 is arcwise connected, that is, *X*2

is connected. In the same way, each *Xi* is convinced to be connected and compact. Applying F. to a compact normal

·**q**

·**p**

**FIGURE 1.** The second step *X*2 of the construction of Sierpin´ski carpet= ∩ *Xi*. Any two points *p* and *q* are joined by an arc in

*X*2.

*i*∈**N**

space [0*,* 1] [0*,* 1], we can see that non-empty compact set *Xi* =SC is connected. There exists a fundamental

∩

×

*i* **N**

∈

difference in topological nature between SC and CMTS. CMTS is 0-dim, i.e, totally disconnected, on the other hand,

SC is connected. The procedure of construction of the self-similar quotient space of CMTS is based on the existence of an adequate partition of CMTS and the existence of this partition is based on the 0-dim of CMTS. Thus, concerning SC, which is connected, no non-trivial quotient space and self-similar, is obtained in an elegant procedure employed for CMTS.

# CONCLUSION

In this study, we have discussed a difference in topological nature between CMTS and SC through the investigations of the procedure of construction of a quotient space and obtained following results concerning the difference in topological nature between CMTS and SC.

1. A topological space, which is homeomorphic to a self-similar space, is self-similar. There exists a non-trivial quotient space *D* of CMTS, which is homeomorphic to CMTS. Since CMTS is self-similar, quotient space *D* is also self-similar.
2. Above result 1) owes mainly to the 0-dim (totally disconnected nature) of the structures of CMTS. Therefore, it is impossible for SC, which is a connected space, to have a self-similar quotient space under the exactly same elegant procedure of construction as that employed for CMTS.

Finally we note that, because of the connectedness of their structures, the same situations with Sierpin´ski carpet exist for Sierpin´ski gasket and Sierpin´ski sponge (Menger sponge).

# ACKNOWLEDGMENTS

The authors are grateful to Professor Emeritus H. Fukaishi at Kagawa University for helpful discussions.

# APPENDIX

1). Proof of the statement in B.

Let a topological space (*X, T* ) be self-similar and homeomorphic to (*Y,* τ). By the above deﬁnition of self-similarity, there exists a metric *d* on *X* such that *T* = τ*d* and there exists a system of contractions

*p j* : (*X,* τ*d* ) → (*X,* τ*d* ), *d*(*pj*(*x*)*, p j*(*x* )) ≤ α *j*(η)*d*(*x, x* ) for *d*(*x, x* ) *<* η*,* 0 ≤ α *j*(η) *<* 1*, j* = 1*,..., m* (2 ≤ *m <* ∞)

*m*

∪ →

satisfying the relation *p j*(*X* ) = *X* . Using a homeomorphism *h* : (*X,* τ*d* ) (*Y,* τ), we can deﬁne a metric ρ on *Y* as

*j*=1

ρ(*y, y* ) = *d*(*h*−1(*y*)*, h*−1(*y* ))*, y, y* ∈ *Y.*

The metric topology τρ is identical with the initial topology τ. From the relations 1) and 2) below, (*Y,* τ)(= (*Y,* τρ )) is self-similar based on a system of contractions *q j* : (*Y,* τρ ) → (*Y,* τρ )*, j* = 1*,..., m* where *q j* is topologically conjugate to *p j* with the above homeomorphism *h*, that is, *q j* = *h* ◦ *p j* ◦ *h*−1 .

1) ρ(*qj*(*y*)*, q j*(*y* )) = *d*(*h*−1(*qj*(*y*))*, h*−1(*qj*(*y* )))

= *d*(*pj*(*h*−1(*y*))*, p j*(*h*−1(*y* ))) α *j*(η)*d*(*h*−1(*y*)*, h*−1(*y* ))

≤

= α *j*(η)ρ(*y, y* ) for ρ(*y, y* ) *<* η.

*m m m*

2) ∪ *q j*(*Y* ) = ∪ *q j*(*h*(*X* )) = *h*( ∪ *p j*(*X* )) = *h*(*X* ) = *Y.* □

*j*=1

*j*=1

*j*=1

2).Proof of the statement in C.

Let *C*(*Z*) and *dH* be the set of all non-empty compact sets of (*Z,* τ*d* ) and the Hausdorff metric, respectively. It is known[7] that the set dynamical system

*m*

∪→ →

*T* : (*C*(*Z*)*,* τ*dH* ) (*C*(*Z*)*,* τ*dH* )*, A f j*(*A*)

*j*=1

*m*

∪⊂

has unique ﬁxed point *X* ; that is, there exists unique non-empty compact set *X* ( *Z*) such that *f j*(*X* ) = *X.*

*j*=1

The condition *iii*) is known[8] as a condition which makes the set *X* 0-dim.

To complete the proof, let us show that the conditions *i*) and *ii*) are sufﬁcient for the subspace (*X,* τ*dX* ) of (*Z,* τ*d* ) (*dX* denotes the restriction of the metric *d* on *X* ) to be perfect. Namely, let us show that any point *x* of *X* is an accumulation

point of *X* . Since any contraction on a complete metric space has one, only one, ﬁxed point[7], it is obvious that the

condition *ii*) requires that *X* is not a singleton. Let *x*0 and *x*0 be different two points of *X* . For any point *x* ∈ *X* and

for any ε *>* 0, let us consider an open sphere *S*(*x,* ε) = {*z* ∈ *Z*; *d*(*x, z*) *<* ε}. The relation ∪

*Xj*1··· *jk* = *X* is valid

for any *k* 1[8]. Here *X*

denotes *f*

*f X j* 1 *m*

*j* 1 *j*1··· *jk*

*X* ≥ *j*1··· *jk*

*j*1 ◦ · · · ◦

*jk* ( )*,* 1 ∈ { *, . . . ,*

}*,...,*

*k* ∈ { *, . . . , m*}, and the diameter of

*j*1··· *jk* → 0 (*k* → ∞)[8]. There exist *k* and a *k*-tuple *j*1 · · · *jk* such that *x* ∈ *Xj*1··· *jk* and the diameter of *Xj*1··· *jk <* ε. Since the contractions *f*1*, . . . , fm* are all one to one from the condition *i*), the point *p* = *f j*1 ◦ · · · ◦ *f jk* (*x*0) ∈ *Xj*1··· *jk* ⊂ *X* and the point *p* = *f j*1 ◦ · · · ◦ *f jk* (*x*0 ) ∈ *Xj*1··· *jk* ⊂ *X* must be different. Therefore, at least either *S*(*x,* ε) − {*x*} *p* or *S*(*x,* ε) −{*x*} *p* holds. This means that the point *x* is an accumulation point of *X* . □

3). Proof of the statement in E.

Since (*Z,* τ) is perfect, (*Z,* τ) has at least two points *z* and *z* . Without loss of generality, there exists *U* τ such that *U* and *z U* . 0-dim guarantees the existence of *u* τ ℑ such that *z u U* . Taking this open and closed set *u Y* , we obtain a 0-dim, perfect, proper subspace (*Y,* τ*Y* ) of (*Z,* τ). □

*z*

as∈ /∈ ∈ ∩ ∈ ⊂

∈

# REFERENCES

1. S.K.Ma, *Modern theory of critical phenomena*, Benjamin (1976). 2. A. Fern*a*´ndez, J. Phys. A **21**, L295 (1988).

1. This dimension is a topological dimension.
2. A.Illanes and S.B.Nadler Jr., *Hyperspace*, Marcel Dekker (1999).
3. The map *h* : (*A,* τ) (*A ,* τ ) is said to be a homeomorphism provided that *h* is one to one, onto and continuous together with the inverse function

→

*h*−1 : (*A ,* τ ) (*A,* τ). Roughly speaking, "the space *A* is homeomorphic to the space *A* " means that *A* and *A* are geometrically equivalent.

→

1. S. B. Nadler, Jr. *Continuum Theory* (Marcel Dekker, New York, 1992).
2. S.Nakamura, T.Konishi and A.Kitada, Journal of the Physical Society of Japan **64** (1995) 731.
3. A.Kitada, T.Konishi, T.Watanabe, Chaos, Solitons & Fractals **13** (2002) 363.